On entropy flux of transversely isotropic elastic bodies

I-Shih Liu*
Instituto de Matemática
Universidade Federal do Rio de Janeiro
21945-970, Rio de Janeiro, Brasil

Abstract

Recently [1] thermodynamic theory of elastic (and viscoelastic) material bodies has been analyzed based on the general entropy inequality. It is proved that for isotropic elastic materials, the results are identical to the classical results based on the Clausius-Duhem inequality [2], for which one of the basic assumptions is that the entropy flux is defined as the heat flux divided by the absolute temperature. For anisotropic elastic materials in general, this classical entropy flux relation has not been proved in the new thermodynamic theory. In this note, as a supplement of the theory presented in [1], it will be proved that the classical entropy flux relation need not be valid in general, by considering a transversely isotropic elastic material body.

Keywords: Clausius-Duhem inequality, general entropy inequality, thermodynamics with Lagrange multiplier, entropy flux relation, transversely isotropic material

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1 Introduction

Exploitation of entropy principle based on the Clausius-Duhem inequality has been widely adopted in the development of modern continuum thermodynamics following the simple Coleman-Noll procedure [2]. The main assumption, that the entropy flux is defined as the heat flux divided by the absolute temperature will be referred to as the *classical entropy flux relation* in this paper. On the other hand, exploitation of the entropy principle based on the general entropy inequality has been proposed by Müller [3] and the method of Lagrange multipliers proposed by Liu [4] greatly facilitates its exploitation procedure.

For isotropic elastic and viscoelastic materials, it is proved in [1] the classical entropy relation remains valid in the new theory. However, for anisotropic elastic materials in general, the validity of the classical entropy flux relation is yet to be explored. In this paper, we shall consider transversely isotropic elastic bodies, and show that the classical entropy flux relation does not hold in general.

2 Thermodynamics of elastic materials

The balance laws of mass, linear momentum, and internal energy,

$$\begin{aligned} &\rho = J^{-1}\rho_\kappa, \\ &\rho_\kappa \ddot{\boldsymbol{x}} - \operatorname{Div} T_\kappa = 0, \\ &\rho_\kappa \dot{\varepsilon} + \operatorname{Div} \boldsymbol{q}_\kappa - T_\kappa \cdot \dot{F} = 0, \end{aligned} \qquad (1)\ (6.1)$$

*e-mail: liu@im.ufrj.br

and the entropy inequality,

$$\rho_\kappa \dot\eta + \operatorname{Div} \boldsymbol{\Phi}_\kappa \geq 0, \tag{2} \quad \text{(6.2)}$$

are given in referential description relative to the reference configuration κ, where the Piola-Kirchhoff stress tensor T_κ and the material heat flux vector $\boldsymbol{q}_\kappa$ are related to the Cauchy stress tensor T and the heat flux vector $\boldsymbol{q}$ by $T_\kappa = JTF^{-T}$ and $\boldsymbol{q}_\kappa = JF^{-1}\boldsymbol{q}$. The material entropy flux $\boldsymbol{\Phi}_\kappa$ is similarly defined. Div is the divergence operator with respect to the referential coordinates, and F is the deformation gradient and $J = |\det F|$.

The constitutive relations for elastic materials can be written as functions of the state variables $(F, \theta, \boldsymbol{g}_\kappa)$,

$$\begin{aligned} T_\kappa &= \widehat{T}_\kappa(F,\theta,\boldsymbol{g}_\kappa), \\ \boldsymbol{q}_\kappa &= \widehat{\boldsymbol{q}}_\kappa(F,\theta,\boldsymbol{g}_\kappa), \\ \varepsilon &= \hat\varepsilon\,(F,\theta,\boldsymbol{g}_\kappa), \end{aligned} \qquad \begin{aligned} \eta &= \hat\eta\,(F,\theta,\boldsymbol{g}_\kappa), \\ \boldsymbol{\Phi}_\kappa &= \widehat{\boldsymbol{\Phi}}_\kappa(F,\theta,\boldsymbol{g}_\kappa). \end{aligned} \tag{3} \quad \text{(6.3)}$$

where $\boldsymbol{g}_\kappa = \nabla\theta$ is the temperature gradient in referential coordinates. Meanwhile, θ will be regarded as an *empirical* temperature, which is some convenient measure of hotness (or coldness) of the thermodynamic state.

For the purpose of determining the constitutive restrictions, the regularities of the constitutive functions as well as the state variables are usually assumed to be smooth enough as the contexts require. No specific regularity requirements will be given in the subsequent discussions.

Summary of thermodynamic restrictions

The entropy principle requires that the entropy inequality (2) be satisfied for any solution of the balance laws (1). Exploitation of this requirement following the Müller-Liu procedure is presented in [1]. We shall summarize restrictions imposed on constitutive equations (3) for elastic material bodies.

For elastic bodies, the *condition of material objectivity* implies the following reduced constitutive equations,

$$\boldsymbol{q}_\kappa = \bar{\boldsymbol{q}}(C,\theta,\boldsymbol{g}_\kappa), \qquad \boldsymbol{\Phi}_\kappa = \bar{\boldsymbol{\Phi}}(C,\theta,\boldsymbol{g}_\kappa), \tag{4} \quad \text{(6.mflux)}$$

where $C = F^TF$ is the right Cauchy-Green tensor, and the entropy principle requires that they must satisfy the relations,

$$\Big(\frac{\partial \bar{\boldsymbol{\Phi}}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big)_{\text{sym}} = \varLambda^\varepsilon \Big(\frac{\partial \bar{\boldsymbol{q}}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big)_{\text{sym}}, \qquad \frac{\partial \bar{\boldsymbol{\Phi}}}{\partial C} - \varLambda^\varepsilon \frac{\partial \bar{\boldsymbol{q}}}{\partial C} = 0, \tag{5} \quad \text{(S1)}$$

where $(A)_{\text{sym}}$ denotes the symmetric part of the tensor A.

The relations involving entropy density can be summarized in the following differential expression, similar to the Gibbs relation in classical thermodynamics,

$$d\eta = \varLambda^\varepsilon \Big(d\varepsilon - \frac{1}{\rho_\kappa} T_\kappa \cdot dF\Big). \tag{6} \quad \text{(S2)}$$

And finally there is a remaining inequality, which gives the entropy production density σ as

$$\sigma = \Big(\frac{\partial \bar{\boldsymbol{\Phi}}_\kappa}{\partial \theta} - \varLambda^\varepsilon \frac{\partial \bar{\boldsymbol{q}}_\kappa}{\partial \theta}\Big) \cdot \boldsymbol{g}_\kappa \geq 0. \tag{7} \quad \text{(S3)}$$

The relations (5), (6), and (7), are the thermodynamic restrictions for elastic materials in general. Note that these relations contain the Lagrange multiplier $\varLambda^\varepsilon$, which also depends on the constitutive variables $(F, \theta, \boldsymbol{g}_\kappa)$. For further evaluations, some specific classes of elastic materials must be considered (see [5]).

Remarks

In [1], it has been proved from (5) that for *isotropic* elastic bodies all the results based on the Clausius-Duhem inequality remain valid. In particular, the Lagrange multiplier Λ^ε can be identified with the reciprocal of the absolute temperature θ and the classical entropy flux relation holds,

$$\Lambda^\varepsilon = \frac{1}{\theta}\,, \qquad \boldsymbol{\Phi}_\kappa = \frac{1}{\theta}\,\boldsymbol{q}_\kappa. \tag{8} \text{(R1)}$$

Moreover, from (6), we have the following Gibbs relation,

$$d\eta = \frac{1}{\theta}\Big(d\varepsilon - \frac{1}{\rho_\kappa}\,T_\kappa\cdot dF\Big). \tag{9} \text{(R2)}$$

It implies that the following constitutive relations are determined completely by the free energy function $\psi(F,\theta)$, defined by $\psi = \varepsilon - \theta\eta$,

$$T_\kappa = \rho_\kappa \frac{\partial\psi}{\partial F}, \qquad \eta = -\frac{\partial\psi}{\partial\theta}, \qquad \varepsilon = \psi - \theta\,\frac{\partial\psi}{\partial\theta}, \tag{10} \text{(R3)}$$

which are usually referred as potential relations, in particular, the first relation defines the body as *hyperelastic*.

3 Transversely isotropic elastic materials

Let $\mathcal{O}$ be the full orthogonal group and $\mathcal{G}_\kappa \subset \mathcal{O}$ be the material symmetric group in the reference configuration κ. Then the material symmetry condition for the material fluxes $\boldsymbol{q}_\kappa$ and $\boldsymbol{\Phi}_\kappa$ can be expressed as

$$\begin{aligned}\bar{\boldsymbol{q}}(QCQ^T,\theta,Q\boldsymbol{g}_\kappa) &= Q\,\bar{\boldsymbol{q}}(C,\theta,\boldsymbol{g}_\kappa),\\ \bar{\boldsymbol{\Phi}}(QCQ^T,\theta,Q\boldsymbol{g}_\kappa) &= Q\,\bar{\boldsymbol{\Phi}}(C,\theta,\boldsymbol{g}_\kappa),\end{aligned} \qquad \forall\, Q\in\mathcal{G}_\kappa. \tag{11} \text{(6.iso)}$$

In other words, the material flux $\boldsymbol{q}_\kappa$ and $\boldsymbol{\Phi}_\kappa$ are invariant functions of $(C,\theta,\boldsymbol{g}_\kappa)$ with respect to the group $\mathcal{G}_\kappa \subset \mathcal{O}$.

We shall consider two different classes of transversely isotropic material bodies, with the following symmetry groups,

$$\begin{aligned}\mathcal{G}_1 &= \{Q\in\mathcal{O}\,|\,Q\boldsymbol{n} = \boldsymbol{n}\},\\ \mathcal{G}_2 &= \{Q\in\mathcal{O}\,|\,Q(\boldsymbol{n}\otimes\boldsymbol{n})Q^T = \boldsymbol{n}\otimes\boldsymbol{n}\},\end{aligned}$$

where $\boldsymbol{n}$ is a fixed unit vector which is the preferred direction of the transverse isotropy of the body. The first symmetry group preserves the direction and the sense of the axis vector $\boldsymbol{n}$, while the second only preserves the axial direction of the vector $\boldsymbol{n}$.

Representations of anisotropic functions with respect to symmetry group of this type have been considered. They can be expressed in terms of isotropic functions from the following theorem (the proof is given in [6, 7]):

Theorem. *A function $f = f(\boldsymbol{v}, A)$ is an invariant function, of vector and tensor variables, with respect to*

1. *symmetry group $\mathcal{G}_1$, if and only if it can be represented by*
   $f(\boldsymbol{v},A) = \widetilde{f}(\boldsymbol{v},A,\boldsymbol{n})$,
2. *symmetry group $\mathcal{G}_2$, if and only if it can be represented by*
   $f(\boldsymbol{v},A) = \widetilde{f}(\boldsymbol{v},A,\boldsymbol{n}\otimes\boldsymbol{n})$,

where $\widetilde{f}$ is an isotropic function with respect to the full orthogonal group $\mathcal{O}$.

In applying isotropic representation for constitutive functions, we shall replace constitutive variable C, the Cauchy-Green strain tensor, by the Green-St.Venant strain tensor E, defined by

$$E = \frac{1}{2}(C - I),$$

which vanishes when there is no deformation and consider constitutive functions of $(E, \theta, \boldsymbol{g}_\kappa)$ up to bilinear terms in E and $\boldsymbol{g}_\kappa$.

3.1 Case 1: Symmetry group $\mathcal{G}_1$

From the above theorem, for a transversely isotropic body with the symmetry group $\mathcal{G}_1$, the constitutive functions for the material heat flux and entropy flux can be represented as isotropic vector functions of $(E, \theta, \boldsymbol{g}_\kappa, \boldsymbol{n})$, one symmetric tensor, one scalar, and two vector variables. The representations can be obtained from the formulas available in the literature (see [7, 8]).

The following are the constitutive equations of material heat flux and entropy flux vectors for a class of transversely isotropic elastic materials with symmetry group $\mathcal{G}_1$. They contain isotropic vector elements $\{\boldsymbol{g}_\kappa, E\boldsymbol{g}_\kappa, \boldsymbol{n}, E\boldsymbol{n}\}$ with coefficients in combinations of isotropic scalar invariants $\{\theta, \operatorname{tr} E, \boldsymbol{n}\cdot E\boldsymbol{n}, \boldsymbol{n}\cdot\boldsymbol{g}_\kappa, \boldsymbol{n}\cdot E\boldsymbol{g}_\kappa\}$ up to bilinear terms in E and $\boldsymbol{g}_\kappa$.

$$\begin{aligned}
\boldsymbol{q}_\kappa &= (a_1 + a_2 \operatorname{tr} E + a_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{g}_\kappa + a_4 E\boldsymbol{g}_\kappa \\
&\quad + (b_1 + b_2 \operatorname{tr} E + b_3\, \boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa + b_4(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa + b_5(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa \\
&\quad + (c_1 + c_2 \operatorname{tr} E + c_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{n} + c_4 E\boldsymbol{n}, \\
\boldsymbol{\Phi}_\kappa &= (\alpha_1 + \alpha_2 \operatorname{tr} E + \alpha_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{g}_\kappa + \alpha_4 E\boldsymbol{g}_\kappa \\
&\quad + (\beta_1 + \beta_2 \operatorname{tr} E + \beta_3\, \boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa + \beta_4(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa + \beta_5(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa \\
&\quad + (\gamma_1 + \gamma_2 \operatorname{tr} E + \gamma_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{n} + \gamma_4 E\boldsymbol{n},
\end{aligned} \qquad (12)\ \text{(C1)}$$

where all the material coefficients are functions of temperature θ only. Note that we have written $(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)\boldsymbol{n}$ as $(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa$, $(\boldsymbol{n}\cdot E\boldsymbol{g}_\kappa)\boldsymbol{n}$ as $(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa$, and $(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)E\boldsymbol{n}$ as $(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa$.

From these representations, we can determine the gradients of the material heat flux with respect to its vector and symmetric tensor variables, $\boldsymbol{g}_\kappa$ and C.

$$\begin{aligned}
\Big(\frac{\partial \boldsymbol{q}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big)_{\text{sym}} &= (a_1 + a_2 \operatorname{tr} E + a_3\, \boldsymbol{n}\cdot E\boldsymbol{n})I + a_4 E \\
&\quad + (b_1 + b_2 \operatorname{tr} E + b_3\, \boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n}) + \frac{1}{2}(b_4 + b_5)(\boldsymbol{n}\otimes E\boldsymbol{n} + E\boldsymbol{n}\otimes\boldsymbol{n}), \\
\frac{\partial(\boldsymbol{v}\cdot\boldsymbol{q}_\kappa)}{\partial C} &= \frac{1}{2}\,(\boldsymbol{v}\cdot\boldsymbol{g}_\kappa)(a_2 I + a_3(\boldsymbol{n}\otimes\boldsymbol{n})) + \frac{1}{4}\,a_4(\boldsymbol{v}\otimes\boldsymbol{g}_\kappa + \boldsymbol{g}_\kappa\otimes\boldsymbol{v}) \\
&\quad + \frac{1}{2}\,(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)(\boldsymbol{v}\cdot\boldsymbol{n})(b_2 I + b_3(\boldsymbol{n}\otimes\boldsymbol{n})) \\
&\quad + \frac{1}{4}\,b_4(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{g}_\kappa + \boldsymbol{g}_\kappa\otimes\boldsymbol{n}) + \frac{1}{4}\,b_5(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)(\boldsymbol{v}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{v}) \\
&\quad + \frac{1}{2}\,(\boldsymbol{v}\cdot\boldsymbol{n})(c_2 I + c_3(\boldsymbol{n}\otimes\boldsymbol{n})) + \frac{1}{4}\,c_4(\boldsymbol{v}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{v}).
\end{aligned}$$

In the second expression, we have introduce an arbitrary constant vector $\boldsymbol{v}$ to reduce the third order gradient tensor $\partial\boldsymbol{g}_\kappa/\partial C$ to a symmetric second order tensor. Corresponding expressions for the entropy flux vector can be similarly written out.

Note that these expressions are isotropic symmetric tensor functions, containing functionally independent symmetric tensor elements $\{I, E, \boldsymbol{n}\otimes\boldsymbol{n}, (\boldsymbol{n}\otimes E\boldsymbol{n} + E\boldsymbol{n}\otimes\boldsymbol{n})\}$ and $\{I, \boldsymbol{n}\otimes\boldsymbol{n}, (\boldsymbol{n}\otimes\boldsymbol{g}_\kappa + \boldsymbol{g}_\kappa\otimes\boldsymbol{n}), (\boldsymbol{v}\otimes\boldsymbol{g}_\kappa + \boldsymbol{g}_\kappa\otimes\boldsymbol{v}), (\boldsymbol{v}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{v})\}$ respectively.

Therefore, after introducing the above gradients into the relation $(5)_1$, the scalar coefficients of the symmetric tensor elements $\{I, E, \boldsymbol{n}\otimes\boldsymbol{n}, (\boldsymbol{n}\otimes E\boldsymbol{n} + E\boldsymbol{n}\otimes\boldsymbol{n})\}$ must vanish,

$$\begin{aligned} \alpha_1 + \alpha_2 \operatorname{tr} E + \alpha_3\, \boldsymbol{n}\cdot E\boldsymbol{n} &= \Lambda^\varepsilon(a_1 + a_2 \operatorname{tr} E + a_3\, \boldsymbol{n}\cdot E\boldsymbol{n}),\\ \beta_1 + \beta_2 \operatorname{tr} E + \beta_3\, \boldsymbol{n}\cdot E\boldsymbol{n} &= \Lambda^\varepsilon(b_1 + b_2 \operatorname{tr} E + b_3\, \boldsymbol{n}\cdot E\boldsymbol{n}),\\ (\beta_4+\beta_5) &= \Lambda^\varepsilon(b_4+b_5),\\ \alpha_4 &= \Lambda^\varepsilon a_4. \end{aligned} \qquad (13)\ \text{(G1)}$$

Note that the left hand sides are independent of the temperature gradient $\boldsymbol{g}_\kappa$, therefore, Λ^ε must be independent of $\boldsymbol{g}_\kappa$. Similarly from $(5)_2$, it gives

$$\begin{aligned} \alpha_2\, \boldsymbol{g}_\kappa + \beta_2\,(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)\boldsymbol{n} + \gamma_2\,\boldsymbol{n} &= \Lambda^\varepsilon(a_2\,\boldsymbol{g}_\kappa + b_2\,(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)\boldsymbol{n} + c_2\,\boldsymbol{n})\\ \alpha_3\, \boldsymbol{g}_\kappa + \beta_3\,(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)\boldsymbol{n} + \gamma_3\,\boldsymbol{n} &= \Lambda^\varepsilon(a_3\,\boldsymbol{g}_\kappa + b_3\,(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa)\boldsymbol{n} + c_3\,\boldsymbol{n})\\ (\beta_5(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa) + \gamma_4) &= \Lambda^\varepsilon(b_5(\boldsymbol{n}\cdot\boldsymbol{g}_\kappa) + c_4),\\ \beta_4 &= \Lambda^\varepsilon b_4, \end{aligned} \qquad (14)\ \text{(G2)}$$

which, likewise, implies that Λ^ε must be independent of the strain tensor E (or C). Therefore, we have proved that the Lagrange multiplier Λ^ε is a function of temperature only.

Consequently, from the entropy relation (6) becomes the classical Gibbs relation (9) by identifying Λ^ε with the absolute temperature,

$$\Lambda^\varepsilon = \Lambda(\theta) = \frac{1}{\theta}\,.$$

and the potential relations (10) remain valid. In particular, the body is hyperelastic.

Furthermore, from $(14)_{1,2,3}$, by evaluated at $\boldsymbol{g}_\kappa = 0$, it follows that

$$\gamma_2 = \frac{1}{\theta}\,c_2, \qquad \gamma_3 = \frac{1}{\theta}\,c_3, \qquad \gamma_4 = \frac{1}{\theta}\,c_4,$$

and from the rest of (13) and (14) we also have

$$\begin{gathered} \alpha_4 = \frac{1}{\theta}\,a_4, \qquad \beta_4 = \frac{1}{\theta}\,b_4, \qquad \beta_5 = \frac{1}{\theta}\,b_5,\\ \alpha_1 + \alpha_2 \operatorname{tr} E + \alpha_3\, \boldsymbol{n}\cdot E\boldsymbol{n} = \frac{1}{\theta}\,(a_1 + a_2 \operatorname{tr} E + a_3\, \boldsymbol{n}\cdot E\boldsymbol{n}),\\ \beta_1 + \beta_2 \operatorname{tr} E + \beta_3\, \boldsymbol{n}\cdot E\boldsymbol{n} = \frac{1}{\theta}\,(b_1 + b_2 \operatorname{tr} E + b_3\, \boldsymbol{n}\cdot E\boldsymbol{n}). \end{gathered}$$

Consequently, from (12), we obtain

$$\boldsymbol{\Phi}_\kappa - \gamma_1\boldsymbol{n} = \frac{1}{\theta}\,(\boldsymbol{q}_\kappa - c_1\boldsymbol{n}).$$

Therefore, the relation between the entropy flux and the heat flux can be expressed as

$$\boldsymbol{\Phi}_\kappa = \frac{1}{\theta}\,\boldsymbol{q}_\kappa + k(\theta)\,\boldsymbol{n}, \qquad (15)\ \text{(F1)}$$

where $k(\theta) = \gamma_1(\theta) - c_1(\theta)/\theta$ and the unit vector $\boldsymbol{n}$ is the preferred direction of the transverse isotropy.

Further results

Further results may follow from the entropy production inequality (7), which becomes

$$\sigma(E,\theta,\boldsymbol{g}_\kappa) = -\frac{1}{\theta^2}\,\boldsymbol{q}_\kappa\cdot\boldsymbol{g}_\kappa + k(\theta)'\,\boldsymbol{n}\cdot\boldsymbol{g}_\kappa \geq 0.$$

The function $\sigma(E,\theta,\boldsymbol{g}_\kappa)$ attains its minimum, namely 0, at $\boldsymbol{g}_\kappa = 0$, therefore by assuming smoothness, it is necessary that

$$\frac{\partial\sigma}{\partial\boldsymbol{g}_\kappa}\Big|_{\boldsymbol{g}_\kappa=0} = -\frac{1}{\theta^2}\,\boldsymbol{q}_\kappa(E,\theta,0) + k(\theta)'\,\boldsymbol{n} = 0,$$

which, by $(12)_1$, leads to

$$(c_1 + c_2\,\mathrm{tr}\,E + c_3\,\boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{n} + c_4 E\boldsymbol{n} - \theta^2 k'\,\boldsymbol{n} = 0.$$

In particular, for $E = 0$, we have

$$k' = \frac{c_1}{\theta^2},$$

and it follows also that

$$(c_2\,\mathrm{tr}\,E + c_3\,\boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{n} + c_4 E\boldsymbol{n} = 0.$$

Moreover, the other necessary condition for minimum requires that the second gradient be positive semi-definite,

$$\frac{\partial^2\sigma}{\partial\boldsymbol{g}_\kappa\partial\boldsymbol{g}_\kappa}\Big|_{\boldsymbol{g}_\kappa=0} \geq 0,$$

from which it follows that the matrix $A = A(E,\theta)$ is negative semi-definite,

$$\begin{aligned} A = {} & (a_1 + a_2\,\mathrm{tr}\,E + a_3\,\boldsymbol{n}\cdot E\boldsymbol{n})I + a_4 E \\ & + (b_1 + b_2\,\mathrm{tr}\,E + b_3\,\boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n}) + \frac{1}{2}(b_4 + b_5)(\boldsymbol{n}\otimes E\boldsymbol{n} + E\boldsymbol{n}\otimes\boldsymbol{n}) \leq 0. \end{aligned}$$

In particular, for $A(0,\theta) = a_1 + b_1(\boldsymbol{n}\otimes\boldsymbol{n})$, it requires that

$$a_1 \leq 0, \qquad a_1 + b_1 \leq 0,$$

which implies that the heat conductivity $\kappa = -a_1$ is non-negative.

In summary, we have the constitutive equations for the heat and the entropy fluxes,

$$\begin{aligned} \boldsymbol{q}_\kappa &= c_1\,\boldsymbol{n} + (a_1 + a_2\,\mathrm{tr}\,E + a_3\,\boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{g}_\kappa + a_4 E\boldsymbol{g}_\kappa \\ &\quad + (b_1 + b_2\,\mathrm{tr}\,E + b_3\,\boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa + b_4(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa + b_5(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa \\ \boldsymbol{\Phi}_\kappa &= \frac{1}{\theta}\,\boldsymbol{q}_\kappa + k(\theta)\,\boldsymbol{n}. \end{aligned}$$

Note that there is a heat flux in the absence of temperature gradient in the direction of the axis of transverse isotropy,

$$\boldsymbol{q}_\kappa(E,\theta,0) = c_1(\theta)\,\boldsymbol{n},$$

and

$$k(\theta) = \int \frac{c_1(\theta)}{\theta^2}\,d\theta$$

may not vanish in general. In this case, the classical entropy flux relation does not hold in general.

3.2 Case 2: Symmetry group $\mathcal{G}_2$

For this class of transversely isotropic bodies, the constitutive equations of material heat flux and entropy flux vectors can be represented as isotropic vector functions of $(E, \theta, \boldsymbol{g}_\kappa, \boldsymbol{n}\otimes\boldsymbol{n})$, two symmetric tensors, one scalar, and one vector variables. Up to bilinear terms in E and $\boldsymbol{g}_\kappa$, they contain isotropic vector elements $\{\boldsymbol{g}_\kappa, E\boldsymbol{g}_\kappa, (\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa, (\boldsymbol{n}\otimes\boldsymbol{n})E\boldsymbol{g}_\kappa, E(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa\}$ with coefficients in combinations of isotropic scalar invariants $\{\theta, \operatorname{tr} E, \operatorname{tr}(E(\boldsymbol{n}\otimes\boldsymbol{n}))\}$.

$$\begin{aligned}
\boldsymbol{q}_\kappa &= (a_1 + a_2 \operatorname{tr} E + a_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{g}_\kappa + a_4 E\boldsymbol{g}_\kappa \\
&\quad + (b_1 + b_2 \operatorname{tr} E + b_3\, \boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa + b_4(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa + b_5(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa \\
\boldsymbol{\Phi}_\kappa &= (\alpha_1 + \alpha_2 \operatorname{tr} E + \alpha_3\, \boldsymbol{n}\cdot E\boldsymbol{n})\boldsymbol{g}_\kappa + \alpha_4 E\boldsymbol{g}_\kappa \\
&\quad + (\beta_1 + \beta_2 \operatorname{tr} E + \beta_3\, \boldsymbol{n}\cdot E\boldsymbol{n})(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa + \beta_4(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa + \beta_5(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa,
\end{aligned} \tag{16}$$ (C2)

where all the material coefficients are functions of temperature θ only. Note that we have written $\operatorname{tr}(E(\boldsymbol{n}\otimes\boldsymbol{n}))$ as $(\boldsymbol{n}\cdot E\boldsymbol{n})$, $(\boldsymbol{n}\otimes\boldsymbol{n})E\boldsymbol{g}_\kappa$ as $(\boldsymbol{n}\otimes E\boldsymbol{n})\boldsymbol{g}_\kappa$, and $E(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa$ as $(E\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{g}_\kappa$.

From the constitutive representations (12) and (16), it is clear that this class of transversely isotropic body is a special case of (12), for which $c_1 = c_2 = c_3 = c_4 = 0$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$. Therefore, we conclude immediately that

$$\Lambda^\varepsilon = \frac{1}{\theta}\,, \qquad \boldsymbol{\Phi}_\kappa = \frac{1}{\theta}\,\boldsymbol{q}_\kappa,$$

and all the classical results remain valid in this case.

4 Final remarks

Although it is well-known that the classical entropy flux relation $\boldsymbol{\Phi}_\kappa = (1/\theta)\,\boldsymbol{q}_\kappa$ is inconsistent with the results from the kinetic theory of ideal gases and is also found to be inappropriate to account for thermodynamics of diffusion, it is often regarded as appropriate for most classical theories of continuum mechanics. However, from thermodynamics based on the general entropy inequality, without preassumption on the entropy flux vector, it is proved, in one of the examples (Case 1) considered in this note, that even for elastic material bodies, the classical entropy flux relation may not hold in general.

In the two examples of transversely isotropic elastic bodies, we have also proved that the body is hyperelastic as a consequence of $\Lambda^\varepsilon = \Lambda(\theta)$, irrespective of whether the classical entropy flux relation is valid or not. To what extent that a (thermo-)elastic material body is hyperelastic remains to be seen.

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