On Well-Posedness of Classical Boundary Conditions in Extended Thermodynamics

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Abstract. The classical well-posed boundary conditions in Navier-Stokes-Fourier (NSF) theory are usually insufficient for the corresponding problems in extended theories of thermodynamics. Some additional boundary data may be needed for the uniqueness of solutions. Owing to the specific structure of systems of balance equations in extended thermodynamics, no such data will be needed in the proposed iterative method by decoupling the system into two subsystems. One of them can be solved uniquely with the classical boundary conditions, and the other determines the remaining non-equilibrium field variables by direct evaluation. The method does not rely on any criterion for uniqueness, in contrast to various criteria proposed for such problems recently.

1 Introduction

Boundary value problems in extended thermodynamics of systems of moment equations usually require more boundary conditions, involving higher order moments, in addition to the well-posed boundary conditions of the corresponding classical Navier-Stokes-Fourier theory. Since the higher order moments do not have clear interpretations in terms of measurable physical quantities, the prescription of their boundary values becomes a problem from the physical point of view.

Mathematically, the uniqueness of solutions of boundary value problems depend on those prescribed values. However, physical experiences seem to suggest that the well-posed boundary conditions of the corresponding problems in classical theory should be sufficient for the uniqueness of the solutions in the extended theories that generalize it. Therefore we call those additional boundary values the uncontrollable boundary data. In molecular or statistical models, those data can be calculated from some assumptions on the boundary at the molecular level. However, in the framework of phenomenological theories, we need some criteria or procedures to determine those boundary data.

It has been suggested by Struchtrup and Weiss [6] that the body itself adjusts those values in such a manner as to minimize its entropy production. This assumption became known as the minimax principle. That principle is capable of determining the uncontrollable boundary data, but it leads, to temperature fields that run counter to physical intuition. Similar principles and other criteria have been proposed elsewhere [1,2,4], but none has been regarded as satisfactory.

All these works seem to suggest that one should be able to solve the problem with the same boundary conditions in the classical theory and an extended theory that generalizes it. In this paper, we shall show that this can be done in extended thermodynamics, *without* relying on any particular criterion, in order to solve boundary value problems supplemented with classical boundary conditions only. The proposed method is an iterative approximation often used to decouple a system of differential equations. We shall explain our basic idea with a simple example.

This is the example considered in [1,2,4–6] – one-dimensional stationary heat conduction between two parallel walls. Specifically, we consider the problem in the 14-moment theory

of a monatomic ideal gas at rest, i.e., the velocity field $\boldsymbol{v} = 0$. In this case, as presented in [4], we have

$$\begin{aligned} &\frac{\mathrm{d}q}{\mathrm{d}x} = 0, \\ &q = -\frac{5}{2}Rp\tau\,\frac{\mathrm{d}\theta}{\mathrm{d}x} - \frac{1}{6}\tau\,\frac{\mathrm{d}\Delta}{\mathrm{d}x}, \\ &\Delta = -28R\tau\,\frac{\mathrm{d}(\theta q)}{\mathrm{d}x}. \end{aligned} \tag{1.1}$$

The equation $(1.1)_1$ is the conservation of energy. The relation $(1.1)_3$ can be regarded as a generalization of Fourier law, with the termal conductivity $\kappa = (5/2)R\tau p$, while $(1.1)_2$ is the additional balance equation for the non-equilibrium fourth order moment Δ in the 14-moment theory. The pressure p is constant, θ is the temperature and τ is the relaxation time. The gas constant is denoted by R (the Boltzmann constant divided by the mass of the molecule).

In the classical Fourier theory of heat conduction, the problem with the following boundary conditions,

$$\theta(0) = \theta_0, \qquad q(L) = q_L, \tag{1.2}$$

is well-posed and it has a unique solution. However, in the 14-moment case, the solution also depends on one additional boundary value, say $\Delta(0)$, for which an improperly assigned value may lead to physically irrelevant results. Additional data of this type are referred to as *uncontrollable* boundary data in [3,4].

Unlike the previous attempts to determine the uncontrollable boundary data so as to have a unique solution by imposing an additional criterion, we shall see that it is possible to obtain a unique solution for the problem with classical boundary conditions only.

We consider the problem consists of the usual conservation laws and the generalized Fourier law with the classical boundary conditions,

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}x} = 0, \\ q = -\kappa\,\dfrac{\mathrm{d}\theta}{\mathrm{d}x} - \dfrac{1}{6}\tau\,\dfrac{\mathrm{d}\Delta}{\mathrm{d}x}, \\ \theta(0) = \theta_0, \qquad q(L) = q_L. \end{cases} \tag{1.3}$$

This boundary value problem can be uniquely solved, once the non-equilibrium moment Δ is given. We can do this by an iterative process. As the initial iterate, we take $\Delta = 0$ at the equilibrium state, and solve the the above problem, then the next iterate of Δ can be calculated from the remaining equation $(1.1)_3$,

$$\Delta = -\frac{56}{5}\frac{\kappa}{p}\,\frac{\mathrm{d}(\theta q)}{\mathrm{d}x}. \tag{1.4}$$

With the new iterate of Δ, the problem (1.3) can be solved again, and the process can be repeated iteratively.

Indeed, with $\Delta = 0$, we obtain the first iterate for θ and q from the problem (1.3). It is exactly the same as the classical Fourier solution,

$$q = q_L, \qquad \theta = \theta_0 - \frac{q_L}{\kappa}\,x. \tag{1.5}$$

Substituting this result into (1.4), we obtain the first iterate of Δ,

$$\Delta = \frac{56}{5}\frac{q_L^2}{p}. \tag{1.6}$$

Since the first iterate of Δ is a constant, its derivative vanishes, and the solution for the second iterate of the problem (1.3) remains unchanged, and so do the subsequent iterates. In other words, the iterative approximation converges trivially and we obtain the solution (1.5) and (1.6) uniquely. In particular, we have also determined the boundary value $\Delta(0)$.

Note that the determination of the uncontrollable boundary value $\Delta(0)$ does not rely on any additional criterion except the assumption of the initial iterate at the equilibrium state. This result has been obtained in [1,4] based on certain criteria.

The iterative procedure to obtain a unique solution in this problem is a trivial one. However, the idea of decoupling the system into two subsystems that can be solved with classical boundary conditions by iterative approximation can easily be adapted to more general problems in extended thermodynamics.

2 An iterative method for decoupled systems

One can always split the system of governing equations in extended thermodynamic theory into two subsystems (I) and (II). System (I) consists of the conservation laws and the equations equivalent to Navier-Stoke law for the stress tensor T and the Fourier law for the heat flux $\boldsymbol{q}$. And System (II) consists of the balance equations for the remaining non-equilibrium field variables, say, $\boldsymbol{M}_a$ for $a = 1, \cdots, m$. More specifically, for boundary value problems – of time-independent processes, System (I) and (II) can be written in the form:

$$\text{(I)} \qquad \begin{cases} \operatorname{div}(\rho\,\boldsymbol{v}) = 0, \\ \operatorname{div}(\rho\,\boldsymbol{v}\otimes\boldsymbol{v} - T) = 0, \\ \operatorname{div}\Big(\rho(\varepsilon + \dfrac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{v})\boldsymbol{v} + \boldsymbol{q} - T\boldsymbol{v}\Big) = 0, \\ T = \hat{T}(\rho, \boldsymbol{v}, \theta, \nabla\rho, \nabla\theta, \nabla\boldsymbol{v}, T, \boldsymbol{q}, \nabla T, \nabla\boldsymbol{q}, \boldsymbol{M}_a, \nabla\boldsymbol{M}_a), \\ \boldsymbol{q} \;= \hat{\boldsymbol{q}}\,(\rho, \boldsymbol{v}, \theta, \nabla\rho, \nabla\theta, \nabla\boldsymbol{v}, T, \boldsymbol{q}, \nabla T, \nabla\boldsymbol{q}, \boldsymbol{M}_a, \nabla\boldsymbol{M}_a), \end{cases}$$

$$\text{(II)} \qquad \begin{cases} \text{for} \quad a = 1, \cdots, m, \\ \boldsymbol{M}_a = \hat{\boldsymbol{M}}_a(\rho, \boldsymbol{v}, \theta, \nabla\rho, \nabla\theta, \nabla\boldsymbol{v}, T, \boldsymbol{q}, \nabla T, \nabla\boldsymbol{q}, \boldsymbol{M}_a, \nabla\boldsymbol{M}_a), \end{cases}$$

owing to the structure of additional balance equations in extended theory of thermodynamics. And according to that, in general, the functions $\hat{T}$, $\hat{\boldsymbol{q}}$, and $\hat{\boldsymbol{M}}_a$ are linear in the gradients and the internal energy ε is a function of (ρ, θ).

We shall decouple System (I) from System(II) by the following iterative scheme so that System (I) becomes a closed system of differential equations for the determination of the fields $(\rho, \boldsymbol{v}, \theta, T, \boldsymbol{q})$ – similar to the classical NSF theory.

$$
\text{(I)}\qquad
\begin{cases}
\operatorname{div}(\overset{(n)}{\rho}\ \overset{(n)}{\boldsymbol{v}}) = 0,\\[2mm]
\operatorname{div}(\overset{(n)}{\rho}\ \overset{(n)}{\boldsymbol{v}}\otimes\overset{(n)}{\boldsymbol{v}} - \overset{(n)}{T}) = 0,\\[2mm]
\operatorname{div}\Big(\overset{(n)}{\rho}\,(\overset{(n)}{\varepsilon} + \frac{1}{2}\overset{(n)}{\boldsymbol{v}}\cdot\overset{(n)}{\boldsymbol{v}})\overset{(n)}{\boldsymbol{v}} + \overset{(n)}{\boldsymbol{q}} - \overset{(n)(n)}{T\,\boldsymbol{v}}\Big) = 0,\\[2mm]
\overset{(n)}{T} = \hat{T}(\overset{(n)}{\rho}, \overset{(n)}{\boldsymbol{v}}, \overset{(n)}{\theta}, \nabla\overset{(n)}{\rho}, \nabla\overset{(n)}{\boldsymbol{v}}, \nabla\overset{(n)}{\theta}, \overset{(n\text{-}1)}{T}, \overset{(n\text{-}1)}{\boldsymbol{q}}, \nabla\overset{(n\text{-}1)}{T}, \nabla\overset{(n\text{-}1)}{\boldsymbol{q}}, \overset{(n\text{-}1)}{\boldsymbol{M}_a}, \nabla\overset{(n\text{-}1)}{\boldsymbol{M}_a}),\\[2mm]
\overset{(n)}{\boldsymbol{q}} = \hat{\boldsymbol{q}}(\overset{(n)}{\rho}, \overset{(n)}{\boldsymbol{v}}, \overset{(n)}{\theta}, \nabla\overset{(n)}{\rho}, \nabla\overset{(n)}{\boldsymbol{v}}, \nabla\overset{(n)}{\theta}, \overset{(n\text{-}1)}{T}, \overset{(n\text{-}1)}{\boldsymbol{q}}, \nabla\overset{(n\text{-}1)}{T}, \nabla\overset{(n\text{-}1)}{\boldsymbol{q}}, \overset{(n\text{-}1)}{\boldsymbol{M}_a}, \nabla\overset{(n\text{-}1)}{\boldsymbol{M}_a}),
\end{cases}
\tag{2.1}
$$

$$
\text{(II)}\qquad
\begin{cases}
\text{for}\quad a = 1, \cdots, m,\\[2mm]
\overset{(n)}{\boldsymbol{M}_a} = \hat{\boldsymbol{M}}_a(\overset{(n)}{\rho}, \overset{(n)}{\boldsymbol{v}}, \overset{(n)}{\theta}, \nabla\overset{(n)}{\rho}, \nabla\overset{(n)}{\boldsymbol{v}}, \nabla\overset{(n)}{\theta}, \overset{(n)}{T}, \overset{(n)}{\boldsymbol{q}}, \nabla\overset{(n)}{T}, \nabla\overset{(n)}{\boldsymbol{q}}, \overset{(n\text{-}1)}{\boldsymbol{M}_a}, \nabla\overset{(n\text{-}1)}{\boldsymbol{M}_a}),\\[2mm]
\overset{(0)}{T_a} = 0, \qquad \overset{(0)}{\boldsymbol{q}_a} = 0, \qquad \overset{(0)}{\boldsymbol{M}_a} = 0.
\end{cases}
\tag{2.2}
$$

Assuming the initial iterate at the equilibrium state would be the most reasonable choice because the validity of the extended moment theory is usually restricted to thermodynamic processes not too far from equilibrium. Furthermore, by initiating the iteration at zero non-equilibrium moments, the first iterate will be exactly the classical NSF solution. Therefore, it is akin to the Maxwellian iteration (also employed in [4]), which is regarded as a method to passing thermodynamic limit from extended thermodynamics to ordinary thermodynamics.

In this iterative scheme, at every iteration $n = 1, 2, 3, \cdots$, the terms involving (n-1)-th iterates in System (I) can be regarded as supply terms in the corresponding system of NSF theory. Therefore, we can assume that it can be solved uniquely for the fields $(\rho, \boldsymbol{v}, \theta, T, \boldsymbol{q})$ with the classical well-posed boundary conditions for NSF theory. Then from System (II), the non-equilibrium fields $\boldsymbol{M}_a$ can be determined by direct substitution. In this manner, classical (NSF) boundary conditions are sufficient to solve boundary value problems in extended thermodynamics without any criterion for the determination of uncontrollable boundary data.

We emphasize that the system of differential equations are valid in the interior points only. However, we shall assume that all fields are smooth up to the boundary so that the boundary value is defined as the one-side limit from the interior points. In other words, the boundary values are not necessary the values at the walls if some slip conditions are assumed.

3 Convergence of iterative approximation

The System (II) can in general be put into a fixed-point problem of the form:

$$\boldsymbol{u} = G\boldsymbol{u}, \qquad \boldsymbol{u} \in X,$$

where G is a non-homogeneous linear differential operator in general on some function space X. The iterative scheme giving in (2.2) can then be written as

$$\boldsymbol{u}_n = G\boldsymbol{u}_{n\text{-}1}, \qquad n = 1, 2, 3\cdots.$$

Unfortunately, it is usually not convergent for differential operators.

In order to circumvent problem of non-convergence, we shall proposed a weaker form of iterative scheme with variable weights given by

$$\boldsymbol{u}_n = w_n G\boldsymbol{u}_{n\text{-}1} + (1 - w_n)\boldsymbol{u}_{n\text{-}1} \quad \text{for} \quad 0 < w_n < 1, \quad n = 1, 2, 3, \cdots. \tag{3.1}$$

It can be rewritten as

$$w_n(G\boldsymbol{u}_{n\text{-}1} - \boldsymbol{u}_{n\text{-}1}) = \boldsymbol{u}_n - \boldsymbol{u}_{n\text{-}1}. \tag{3.2}$$

Note that in practice, we shall take the weights w_n monotonically decreasing in n, e.g., $w_n = 1/n$, such that

$$\lim_{n\to\infty} w_n = 0.$$

Therefore, from (3.2), it is obvious that the convergence of the scheme, i.e., $\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u}^*$ is not sufficient to guarantee that $\boldsymbol{u}^*$ is a fixed point of the operator G, because (3.2) is satisfied irrespective of whether $G\boldsymbol{u}^* = \boldsymbol{u}^*$ or not.

However, from (3.2), we can define the *error estimate* of the fixed point of G at the n-th approximation in L^2-norm by

$$\mathrm{Er}\,(n) = \frac{\|\boldsymbol{u}_n - \boldsymbol{u}_{n\text{-}1}\|}{w_n}, \tag{3.3}$$

which estimates the error of how the approximate solution satisfies the equation $G\boldsymbol{u} = \boldsymbol{u}$. Accordingly, the convergence of the scheme for the fixed-point problem is to require that the following condition be verified,

$$\lim_{n\to\infty} \mathrm{Er}\,(n) = \lim_{n\to\infty} \|G\boldsymbol{u}_{n\text{-}1} - \boldsymbol{u}_{n\text{-}1}\| = 0,$$

so that

$$\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u}^* \qquad \text{and} \qquad G\boldsymbol{u}^* = \boldsymbol{u}^*.$$

Therefore, we can state the following convergence criterion.

The variable weighted iterative scheme

$$\boldsymbol{u}_n = w_n G\boldsymbol{u}_{n\text{-}1} + (1 - w_n)\boldsymbol{u}_{n\text{-}1}, \quad w_n = \frac{1}{n}, \quad n = 1, 2, 3, \cdots,$$

with initial iterate $\boldsymbol{u}_0$, converges to a fixed point of $\boldsymbol{u} = G\boldsymbol{u}$, if and only if

$$\lim_{n\to\infty} \mathrm{Er}\,(n) = \frac{\|\boldsymbol{u}_n - \boldsymbol{u}_{n\text{-}1}\|}{w_n} = 0.$$

The convergence of the iterative approximation clearly depend on the property of the operator G. The characterization of the operator G for convergence is under investigation. Nevertheless, we shall consider a case of interest in the next section to illustrate the proposed iterative method by numerically showing that, for small enough boundary data, the limit of $\mathrm{Er}\,(n)$ tends to zero as n increases, therefore the convergence can be justified.

4 Example: heat conduction between coaxial cylinders

The corresponding problem of the introductory example in cylindrical coordinates (r, ϑ, z) is a nontrivial example to illustrate the proposed method of solution.

We consider the boundary value problem of heat conduction in a gas at rest between two coaxial cylinders, with radii $r_1 > r_0$. In the 14-moment theory, we have the following basic equations (see [4]):

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}r} + \dfrac{1}{r}\,q = 0, \\ q = -\dfrac{5}{2}\,\tau R p \dfrac{\mathrm{d}\theta}{\mathrm{d}r} - \dfrac{1}{6}\,\tau \dfrac{\mathrm{d}\Delta}{\mathrm{d}r} + \dfrac{7}{2}\,\tau R\, T^{\langle rr\rangle} \dfrac{\mathrm{d}\theta}{\mathrm{d}r}, \\ T^{\langle rr\rangle} = \dfrac{4}{5}\,\tau\,\dfrac{\mathrm{d}q}{\mathrm{d}r}, \qquad T^{\langle\vartheta\vartheta\rangle} = \dfrac{4}{5}\,\tau \dfrac{1}{r^3}\,q, \qquad \Delta = -28\tau R q \dfrac{\mathrm{d}\theta}{\mathrm{d}r}, \end{cases} \quad \text{for} \quad r_0 \le r \le r_1, \tag{4.1}$$

and the boundary conditions,

$$\theta(r_1) = \theta_1, \qquad q(r_0) = q_0, \tag{4.2}$$

where $T^{\langle ij\rangle}$ is the deviatoric stress tensor and the pressure p is constant. It has been pointed out that the boundary conditions (4.2), which are well-posed in the Fourier theory - consisting of equations $(4.1)_{1,2}$ with $\Delta = T^{\langle rr\rangle} = 0$, are not sufficient for the uniqueness of the above problem in 14-moment theory, and indeed, the additional uncontrollable boundary value, say, $\Delta(r_0) = \Delta_0$ is needed.

We introduce the following dimensionless quantities,

$$\begin{aligned} &r' = \frac{r}{r_1}, && \theta' = \frac{\theta}{\theta_1}, && q' = \frac{q}{p\sqrt{R\theta_1}}, \\ &T^{\langle rr\rangle\prime} = \frac{T^{\langle rr\rangle}}{p}, && T^{\langle\vartheta\vartheta\rangle\prime} = \frac{T^{\langle\vartheta\vartheta\rangle}}{p/r_1^2}, && \Delta' = \frac{\Delta}{pR\theta_1}, \end{aligned}$$

and the Knudsen number,

$$\mathtt{Kn} = \frac{\tau}{r_1}\sqrt{R\theta_1}.$$

4.1 Iterative scheme with variable weights

We shall split the system (4.1), in dimensionless form without the primes for simplicity, into two subsystems – (I) corresponds to the Fourier theory, with the iterative scheme:

$$\text{(I)} \qquad \begin{cases} \dfrac{\mathrm{d}\overset{(n)}{q}}{\mathrm{d}r} + \dfrac{1}{r}\overset{(n)}{q} = 0, \\ \overset{(n)}{q} = -\dfrac{5}{2}\,\mathtt{Kn}\dfrac{\mathrm{d}\overset{(n)}{\theta}}{\mathrm{d}r} - \dfrac{1}{6}\,\mathtt{Kn}\dfrac{\mathrm{d}\overset{(n\text{-}1)}{\Delta}}{\mathrm{d}r} + \dfrac{7}{2}\,\mathtt{Kn}\,\overset{(n\text{-}1)}{T}{}^{\langle rr\rangle}\dfrac{\mathrm{d}\overset{(n)}{\theta}}{\mathrm{d}r}, \end{cases} \tag{4.3}$$

$$\text{(II)} \qquad \begin{cases} \overset{(n)}{T}{}^{\langle rr\rangle} = w_n\Big(\dfrac{4}{5}\,\mathtt{Kn}\dfrac{\mathrm{d}\overset{(n)}{q}}{\mathrm{d}r}\Big) + (1-w_n)\overset{(n\text{-}1)}{T}{}^{\langle rr\rangle}, & \overset{(0)}{T}{}^{\langle rr\rangle} = 0, \\ \overset{(n)}{T}{}^{\langle\vartheta\vartheta\rangle} = w_n\Big(\dfrac{4}{5}\,\mathtt{Kn}\dfrac{\overset{(n)}{q}}{r^3}\Big) + (1-w_n)\overset{(n\text{-}1)}{T}{}^{\langle\vartheta\vartheta\rangle}, & \overset{(0)}{T}{}^{\langle\vartheta\vartheta\rangle} = 0, \\ \overset{(n)}{\Delta} = w_n\Big(-28\,\mathtt{Kn}\,\overset{(n)}{q}\dfrac{\mathrm{d}\overset{(n)}{\theta}}{\mathrm{d}r}\Big) + (1-w_n)\overset{(n\text{-}1)}{\Delta}, & \overset{(0)}{\Delta} = 0. \end{cases} \tag{4.4}$$

The iterative method proceeds alternatively between solving System (I) for $\overset{(n)}{\theta}$ and $\overset{(n)}{q}$ with the (classical) boundary conditions,

$$\overset{(n)}{\theta}(1) = 1, \qquad \overset{(n)}{q}(a) = q_a \quad \text{for} \quad a = r_0/r_1 < 1, \tag{4.5}$$

and then evaluating System (II) for $\overset{(n)}{T}{}^{\langle rr\rangle}$, $\overset{(n)}{T}{}^{\langle \vartheta\vartheta\rangle}$, and $\overset{(n)}{\Delta}$. The iterative process can be repeated with $n = 1, 2, 3, \cdots$ and the error estimate $\mathrm{Er}\,(n)$ can be verified for convergence by checking if it tends to zero as n increases.

4.2 Numerical examples

Note that from $(4.3)_1$, one can solve for $\overset{(n)}{q}$ independent of n with the boundary condition $(4.5)_2$,

$$\overset{(n)}{q}\,(r) = \frac{a\,q_a}{r} \qquad \forall\ n = 1, 2, 3, \cdots.$$

By taking $w_n = 1/n$, from $(4.4)_{1,2}$, it follows that

$$\overset{(n)}{T}{}^{\langle rr\rangle} = -\frac{4}{5}\,\mathtt{Kn}\,\frac{a\,q_a}{r^2}, \qquad \overset{(n)}{T}{}^{\langle \vartheta\vartheta\rangle} = \frac{4}{5}\,\mathtt{Kn}\,\frac{a\,q_a}{r^4}, \qquad \forall\ n = 1, 2, 3, \cdots.$$

Therefore, only $\overset{(n)}{\theta}$ and $\overset{(n)}{\Delta}$ are to be solved iteratively.

For $n = 1$, System (I) is the classical problem of Fourier theory and can be solved analytically with boundary condition $(4.5)_1$,

$$\overset{(1)}{\theta}\,(r) = 1 - \frac{2}{5}\,\frac{a\,q_a}{\mathtt{Kn}}\,\log r, \tag{4.6}$$

and for System (II) we obtain

$$\overset{(1)}{\Delta}(r) = \frac{56}{5}\,\Big(\frac{a\,q_a}{r}\Big)^2. \tag{4.7}$$

Further iterations involve only integrations and derivations and, in principle, can also be solved analytically. However, in order to show the convergence of the error estimate for large n, we shall solve the problem numerically, for some given values of $\mathtt{Kn}$, a, q_a.

In this case, from (3.3), the error estimate $\mathrm{Er}\,(n)$ will be defined as

$$\mathrm{Er}\,(n) = \|\overset{(n)}{\Delta} - \overset{(n\text{-}1)}{\Delta}\,\| \,/\, w_n, \tag{4.8}$$

and in order to show the relative size of the error, we shall also introduce the *relative* error estimate $\mathrm{Err}\,(n)$ defined by

$$\mathrm{Err}\,(n) = \mathrm{Er}\,(n) \,/\, \|\overset{(n)}{\Delta}\|, \tag{4.9}$$

where the norm is defined as

$$\|f\| = \Big\{2\pi \int_a^1 |f(r)|^2\, r dr\Big\}^{1/2},$$

and will be calculated numerically with a simple scheme such as Trapezoidal rule.

Example 1: $\mathtt{Kn} = 0.05$, $a = 0.4$, $q_a = 0.3$, and $w_n = 1/n$.

$\mathrm{Err}\,(2) = 8.55\%$ $\mathrm{Err}\,(10) = 0.60\%$ $\mathrm{Err}\,(100) = 0.040\%$ $\mathrm{Err}\,(1000) = 0.0031\%$

$\theta(a) \approx 1.8188$ $\Delta(a) \approx 0.8690$ $\Delta(1) \approx 0.1570$

Example 2: $\mathtt{Kn} = 0.1$, $a = 0.4$, $q_a = 0.15$, and $w_n = 1/n$.

$\mathrm{Err}\,(2) = 8.55\%$ $\mathrm{Err}\,(10) = 0.60\%$ $\mathrm{Err}\,(100) = 0.040\%$ $\mathrm{Err}\,(1000) = 0.0031\%$

$\theta(a) \approx 1.2047$ $\Delta(a) \approx 0.2172$ $\Delta(1) \approx 0.0393$

Example 3: $\mathtt{Kn} = 0.15$, $a = 0.2$, $q_a = 0.1$, and $w_n = 1/n$.

$\mathrm{Err}\,(2) = 15.40\%$ $\mathrm{Err}\,(10) = 0.78\%$ $\mathrm{Err}\,(100) = 0.046\%$ $\mathrm{Err}\,(1000) = 0.0033\%$

$\theta(a) \approx 1.0789$ $\Delta(a) \approx 0.0865$ $\Delta(1) \approx 0.0044$

Example 4: $\mathtt{Kn} = 0.0832$, $a = 0.3333$, $q_a = 0.4005$, and $w_n = 1/n$.

$\mathrm{Err}\,(2) = 20.95\%$ $\mathrm{Err}\,(10) = 0.89\%$ $\mathrm{Err}\,(100) = 0.054\%$ $\mathrm{Err}\,(1000) = 0.0045\%$

$\theta(a) \approx 1.6073$ $\Delta(a) \approx 1.2941$ $\Delta(1) \approx 0.1900$

The data in this example are chosen for comparison corresponding to the numerical values given in the example of Sect. 5.3 of [4]. The physical values of the reference quantities are given by $R = 207.83\,(\mathrm{m/s})^2\mathrm{K}^{-1}$ for argon gas, $\tau = 10^{-5}\,\mathrm{s}$, $\theta_1 = 300\,\mathrm{K}$, $p = 100\,\mathrm{Pa}$, $r_0 = 0.01\,\mathrm{m}$, $r_1 = 0.03\,\mathrm{m}$, and $q_0 = 10^4\,\mathrm{Wm}^{-2}$. The physical values for the temperature θ and the non-equilibrium moment Δ at the boundary obtained from this example can be calculated.

$\theta(r_0) \approx 482.19\,\mathrm{K}$ $\Delta(r_0) \approx 8.0686\mathrm{x}10^6\,\mathrm{Ns}^{-2}$ $\Delta(r_1) \approx 1.1846\mathrm{x}10^6\,\mathrm{Ns}^{-2}$

There are in perfect agreement with the values obtained in [4], $\theta(r_0) = 482.76\,\mathrm{K}$ and $\Delta(r_1) = 1.1833\mathrm{x}10^6\,\mathrm{Ns}^{-2}$.

4.3 Remarks

The examples show that for small enough boundary data and various Knudsen number, the convergence of the proposed iterative approximation can be achieved with properly chosen monotonically decreasing weights w_n. The evidences of convergence are confirmed numerically from the relative error estimates, which monotonically decrease toward zero as iteration increases. A typical behavior of $\mathrm{Err}\,(n)$ vs. n is plotted in Fig. 1.

From Example 1 and 2, we can see that the behavior of $\mathrm{Err}\,(n)$ for given radius ratio a depends on the product $(\mathtt{Kn} \cdot q_a)$, which has been confirmed from our preliminary analysis of the corresponding operator G. The first three examples converge to within an error of 1.0% in less than 10 iterations.

For all examples considered, the solutions for the temperature $\theta(r)$ are qualitatively similar to the solution (4.6) of the Fourier theory, with greater difference where the temperature gradient is larger (see the comparison in Fig. 3 of [4]). Similar behavior can be said about the non-equilibrium moment $\Delta(r)$ with respect to the solution (4.7). This is what we expected since the 14-moment theory is regarded as a slight improvement of the classical Navier-Stokes-Fourier theory.

The numerical results illustrated in these examples, show that the proposed iterative method is capable of determining a unique solution with only classical well-posed boundary conditions, without relying on any particular criterion. The method has also been applied to shearing flows with heat conduction in extended thermodynamics with equally satisfactory results. Those results and some other comments on the convergence will be forthcoming.

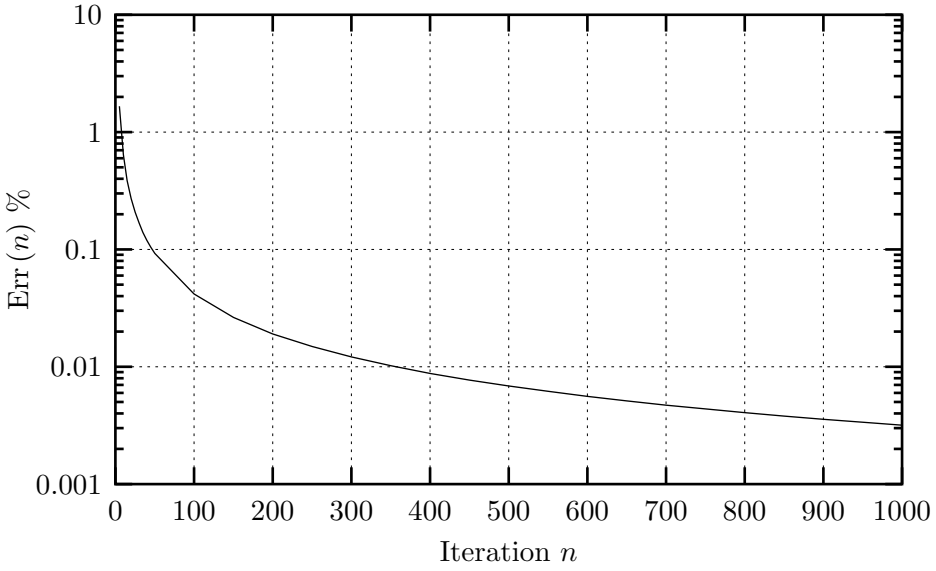

Fig. 1. Semi-log plot of Err (n) vs. n for Example 1

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