Constitutive theory of anisotropic rigid heat conductors

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Abstract

For rigid heat conductors, the constitutive equation of the heat flux is usually assumed to be independent of the deformation of the body, or equivalently by setting the deformation gradient as the identity tensor in the constitutive equations. However, the notion of material symmetry would have involved change of reference configurations, hence also change of deformation gradients. Consequently it would be impossible to consider anisotropic rigid heat conductors from such a formulation. In this paper, rigid heat conductors are treated with rigidity as internal constraint and a constitutive theory consistent with the principle of material frame indifference, material symmetry, and the entropy principle is presented. Constitutive restrictions for a transversely isotropic rigid heat conductor are carefully analyzed.

1 Introduction

In constitutive theories, rigid bodies are regarded as material bodies capable of undergoing rigid body motions only, for which the deformation gradient is an orthogonal transformation. The body does not suffer any mechanical strain and the stress tensor is completely indeterminable from constitutive variables. Therefore, for rigid heat conductors, it is usually assumed that the heat flux and the internal energy density are independent of deformation gradient, and depend only on thermal variables, the temperature and temperature gradient. However, it is inappropriate to formulate in this manner, because it is then impossible to impose constitutive restrictions from material frame indifference and material symmetry. Since neglecting deformation gradient is equivalent to assuming the deformation gradient as an identity tensor, and hence, it is impossible to make change of either observer or reference configuration. As a consequence, anisotropic rigid heat conductors cannot be formulated.

Take for example, if one proposes a model of rigid heat conductor by postulating that the heat flux $\boldsymbol{q}$ is given by the constitutive equation of the form,

$$\boldsymbol{q} = \boldsymbol{q}(\theta, \boldsymbol{g}),$$

where $\boldsymbol{g}$ is the spatial gradient of the temperature θ. Since both $\boldsymbol{q}$ and $\boldsymbol{g}$ are objective vectors, the principle of material frame indifference will require that

$$\boldsymbol{q}(\theta, Q\boldsymbol{g}) = Q\,\boldsymbol{q}(\theta, \boldsymbol{g})$$

for any orthogonal transformation Q. This relation states that the heat flux $\boldsymbol{q}$ is an isotropic vector function of the variable $(\theta, \boldsymbol{g})$. In other words, the proposed constitutive model is necessary a material model for isotropic rigid heat conductors.

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Unlike such simple-minded formulations, we shall correctly formulate a constitutive theory for rigid heat conductors consistent with the following basic requirements ([6, 11]):

- rigid body motion as internal mechanical constraint,
- principle of material frame indifference,
- material symmetry,
- entropy principle.

Remark. The present work is motivated by a suggestion in attempting to consider anisotropic rigid heat conductors from the theories formulated in [1, 2] in the framework of Extended Thermodynamics. In such theories, the heat flux vector is regarded as a field variable, just as the mass density and the linear momentum are, and is not longer regarded as a constitutive quantity. Moreover, an additional field equation is postulated to govern the rate of change of the heat flux. In this equation, the "flux" of the heat flux is regarded as a tensorial constitutive quantity. While the field equations are required to be Galilean invariant, the constitutive equations are required to satisfy the above requirements as usual, among them, the principle of material frame indifference and the material symmetry (see [9, 10]). We remark that considerations for the heat flux vector presented in Section 2. and 3. can easily be generalized to constitutive quantities of any tensorial order under rigidity constraint.

2 Rigidity as internal constraint

Let $\kappa : \mathcal{B} \to \mathbb{E}$ be a reference configuration of the body $\mathcal{B}$ in a Euclidean space $\mathbb{E}$ with the associated vector space V, and

$$\boldsymbol{x} = \boldsymbol{x}(X, t), \qquad X \in \kappa(B),$$

be a motion, and $F = \nabla_X \boldsymbol{x}(X, t)$ be the deformation gradient. A rigid body motion is defined as

$$\boldsymbol{x}(X, t) = Q(t)(X - X_0) + c(t),$$

where $Q(t) \in \mathcal{O}$ and $X_0, c(t) \in \mathbb{E}$; hence the deformation gradient $F(X, t) = Q(t)$ is an orthogonal transformation. Note that orthogonal transformations preserve inner product and hence preserve the geometric shape of the body, so that the Cauchy-Green strain tensors, $C = F^T F$ and $B = FF^T$ are identity tensors. We have denote the group of orthogonal transformations by $\mathcal{O}$.

A rigid body is regarded as a material body with rigidity as internal constraint. From mechanical respect, it is assumed that in order to maintain rigidity in the motions of the body, a reaction stress N which does not work is needed, i.e., the stress tensor

$$T(X, t) = N(X, t) + S(X, t),$$

where S is the extra stress which is a constitutive quantity subject to general constitutive requirements, and $N \cdot D = 0$ for any rate of strain tensor D, the symmetric part of the velocity gradient.

Since for rigid motions, F is orthogonal, by taking the time derivative of $FF^T = I$, we have

$$\dot{F}F^T + F\dot{F}^T = \operatorname{grad} \dot{\boldsymbol{x}} + \operatorname{grad} \dot{\boldsymbol{x}}^T = 0,$$

where we have used $F^T = F^{-1}$ and $\operatorname{grad} \dot{\boldsymbol{x}} = \dot{F}F^{-1}$. Therefore, the rate of strain $D = 0$ identically in the motion, and the condition $N \cdot D = 0$ implies that the reaction stress N is arbitrary. Consequently, the total stress T is indeterminable by the deformations and can only be determined from the equation of motion and boundary conditions of the body.

On the other hand, for thermally related constitutive quantities, the heat flux $\boldsymbol{q}$ and the internal energy density ε, the constitutive functions are to be determined from thermomechanical variables consistent with rigid body motions.

3 Rigid heat conductors

To be more specific, we shall consider a material model with constitutive equations of the form,

$$\boldsymbol{q} = \boldsymbol{q}(F, \theta, \boldsymbol{g}), \qquad \varepsilon = \varepsilon(F, \theta, \boldsymbol{g}),$$

subject to the restrictions that the deformation gradient F must be orthogonal.

The constitutive functions are required to satisfy the condition of material frame indifference,

$$\begin{aligned} \boldsymbol{q}(QF, \theta, Q\boldsymbol{g}) &= Q\boldsymbol{q}(F, \theta, \boldsymbol{g}), \\ \varepsilon(QF, \theta, Q\boldsymbol{g}) &= \varepsilon(F, \theta, \boldsymbol{g}), \end{aligned} \qquad \forall\, Q \in \mathcal{O}, \tag{1}$$

and the condition of material symmetry,

$$\begin{aligned} \boldsymbol{q}(FG, \theta, \boldsymbol{g}) &= \boldsymbol{q}(F, \theta, \boldsymbol{g}), \\ \varepsilon(FG, \theta, \boldsymbol{g}) &= \varepsilon(F, \theta, \boldsymbol{g}), \end{aligned} \qquad \forall\, G \in \mathcal{G}_\kappa, \tag{2}$$

where $\mathcal{G}_\kappa$ is the material symmetry group in the reference configuration κ of the body. For solid rigid heat conductors, the symmetry group $\mathcal{G}_\kappa$ is a subgroup of the orthogonal group $\mathcal{O}$. Note that these conditions are valid for any rigid deformation, so that by polar decomposition $F = RU$, the stretch tensor U is an identity tensors and $F = R$. Therefore, they are valid for any rotation tensor R of the deformation.

As consequences of these requirements, from (1), by taking $Q = F^T \in \mathcal{O}$, we have

$$\boldsymbol{q}(I, \theta, F^T\boldsymbol{g}) = F^T\boldsymbol{q}(F, \theta, \boldsymbol{g}). \tag{3}$$

Furthermore, for rigid solid bodies, we have $\mathcal{G}_\kappa \subseteq \mathcal{O}$ and from (2), by taking $F = G^T$ for $G \in \mathcal{G}_\kappa$, we have

$$\boldsymbol{q}(I, \theta, \boldsymbol{g}) = \boldsymbol{q}(G^T, \theta, \boldsymbol{g}) = G^T\boldsymbol{q}(I, \theta, G\boldsymbol{g}), \tag{4}$$

in the last passage (3) is used. If we define

$$\hat{\boldsymbol{q}}(\theta, \boldsymbol{g}) = \boldsymbol{q}(I, \theta, \boldsymbol{g}),$$

from (3), the constitutive equation for the heat flux is given by

$$\boldsymbol{q} = \boldsymbol{q}(F, \theta, \boldsymbol{g}) = F\hat{\boldsymbol{q}}(\theta, F^T\boldsymbol{g}) \qquad \forall\, F \in \mathcal{O}, \quad \forall\, \boldsymbol{g} \in V, \tag{5}$$

and the relation (4) becomes

$$\hat{\boldsymbol{q}}(\theta, G\boldsymbol{g}) = G\hat{\boldsymbol{q}}(\theta, \boldsymbol{g}), \qquad \forall\, G \in \mathcal{G}_\kappa \subseteq \mathcal{O}, \quad \forall\, \boldsymbol{g} \in V. \tag{6}$$

Similarly, for the internal energy, we have

$$\varepsilon = \varepsilon(F, \theta, \boldsymbol{g}) = \hat{\varepsilon}(\theta, F^T\boldsymbol{g}) \qquad \forall\, F \in \mathcal{O}, \quad \forall\, \boldsymbol{g} \in V, \tag{7}$$

and

$$\hat{\varepsilon}(\theta, G\boldsymbol{g}) = \hat{\varepsilon}(\theta, \boldsymbol{g}), \qquad \forall\, G \in \mathcal{G}_\kappa \subseteq \mathcal{O}, \quad \forall\, \boldsymbol{g} \in V. \tag{8}$$

The conditions (6) and (8) state that the function $\hat{\boldsymbol{q}}$ and $\hat{\varepsilon}$ are invariant functions with respect to the symmetry group of the body. In particular, if the body is isotropic, they are isotropic functions.

For rigid bodies, the internal energy equation in the spatial description,

$$\rho\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} - \operatorname{tr} T\cdot \operatorname{grad}\dot{\boldsymbol{x}} = \rho r,$$

reduces to

$$\rho\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} = \rho r, \tag{9}$$

since $\operatorname{grad}\dot{\boldsymbol{x}} = \dot{F}F^{-1}$ is a skew symmetric tensor for any orthogonal tensor F, and hence $\operatorname{tr} T\cdot \operatorname{grad}\dot{\boldsymbol{x}} = 0$.

Since for rigid body, the deformation gradient F reduces to its rotation part R, from (5) and (7), the constitutive equations can be written as

$$\boldsymbol{q} = R\,\hat{\boldsymbol{q}}(\theta, R^T\boldsymbol{g}), \qquad \varepsilon = \hat{\varepsilon}(\theta, R^T\boldsymbol{g}). \tag{10}$$

These constitutive equations are expected because even though the body does not suffer any mechanical strain, it does undergo rotation in the motion. Note that with these constitutive equations the energy equation (9) contains the rotation tensor R of the motion. Therefore, it is inconvenient to use the energy equation in the spatial description for heat conduction.

We can rewrite the energy equation in the referential description,

$$\rho_\kappa\dot{\varepsilon}_\kappa + \operatorname{Div}\boldsymbol{q}_\kappa = \rho_\kappa r. \tag{11}$$

where Div is the divergence with respect to the referential configuration and

$$\boldsymbol{q}_\kappa = |\det F|F^T\boldsymbol{q}, \qquad \varepsilon_\kappa = |\det F|\varepsilon, \qquad \rho_\kappa = |\det F|\rho.$$

Since F is orthogonal, $|\det F| = 1$ and $F = R$ by polar decomposition, it follows from (10) that

$$\boldsymbol{q}_\kappa = \boldsymbol{q}_\kappa(\theta, \boldsymbol{g}_\kappa), \qquad \varepsilon_\kappa = \varepsilon_\kappa(\theta, \boldsymbol{g}_\kappa), \tag{12}$$

where $\boldsymbol{g}_\kappa = F^T\boldsymbol{g}$ is the referential gradient of temperature, and for convenience, we have written $\boldsymbol{q}_\kappa$ and ε_κ for the functions $\hat{\boldsymbol{q}}$ and $\hat{\varepsilon}$. From the conditions (6) and (8), the constitutive functions $\boldsymbol{q}_\kappa$ and ε_κ are invariant vector and scalar functions with respect to the symmetry group respectively,

$$\begin{aligned} \boldsymbol{q}_\kappa(\theta, G\boldsymbol{g}_\kappa) &= G\,\boldsymbol{q}_\kappa(\theta, \boldsymbol{g}_\kappa),\\ \varepsilon_\kappa(\theta, G\boldsymbol{g}_\kappa) &= \varepsilon_\kappa(\theta, \boldsymbol{g}_\kappa), \end{aligned} \qquad \forall\, G \in \mathcal{G}_\kappa, \quad \forall\, \boldsymbol{g}_\kappa \in V. \tag{13}$$

Remark. For rigid heat conductors, the equations (11), (12), and (13) are usually postulated simply by intuition. The formulation in the referential description is sometimes overlooked, namely, instead of (11) and (12), they are written as

$$\rho\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} = \rho r,$$

and

$$\boldsymbol{q} = \boldsymbol{q}(\theta, \boldsymbol{g}), \qquad \varepsilon = \varepsilon(\theta, \boldsymbol{g}).$$

For rigid bodies, since rigid motions are irrelevant in heat conduction, the bodies are assumed to be at rest, and hence there is no difference between the spacial and the referential descriptions. However, oftentimes, this simple-minded formulation may cause some confusions and may lead to wrong conclusions regarding material frame-indifference and material symmetry requirements as we have pointed out in the Introduction.

4 Thermodynamic restrictions

Since external sources are irrelevant to intrinsic properties of material bodies, for analyzing thermodynamic restrictions of constitutive functions, we shall consider only supply-free bodies, so that we have the energy equation,

$$\rho_\kappa \dot{\varepsilon}_\kappa + \operatorname{Div} \boldsymbol{q}_\kappa = 0,$$

and the entropy inequality,

$$\rho_\kappa \dot{\eta}_\kappa + \operatorname{Div} \boldsymbol{\Phi}_\kappa \geq 0,$$

where η_κ and $\boldsymbol{\Phi}_\kappa$ are the entropy density and the entropy flux in the reference configuration, given by constitutive equations of the form (12).

After introducing the constitutive equations (12) into the energy equation, it becomes a partial differential equation of the temperature field $\theta(X,t)$. Any solution of the field equation will be called a thermodynamic process. It is postulated that the entropy principle requires the entropy inequality to be consistent with any thermodynamic process. The principle is equivalent to require the following inequality (see [3, 6])

$$\rho_\kappa \dot{\eta}_\kappa + \operatorname{Div} \boldsymbol{\Phi}_\kappa - \Lambda \left(\rho_\kappa \dot{\varepsilon}_\kappa + \operatorname{Div} \boldsymbol{q}_\kappa \right) \geq 0, \tag{14}$$

where $\Lambda(\theta, \boldsymbol{g}_\kappa)$, referred to as the Lagrange multiplier, must hold for any smooth field $\theta(X,t)$, instead of thermodynamic fields only.

By the constitutive equations of the form (12), the equality (14) becomes

$$\begin{aligned} \rho_\kappa \Big(\frac{\partial \eta_\kappa}{\partial \theta} - \Lambda \frac{\partial \varepsilon_\kappa}{\partial \theta} \Big) \dot{\theta} + \rho_\kappa \Big(\frac{\partial \eta_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda \frac{\partial \varepsilon_\kappa}{\partial \boldsymbol{g}_\kappa} \Big) \cdot \dot{\boldsymbol{g}}_\kappa \\ + \Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \theta} - \Lambda \frac{\partial \boldsymbol{q}_\kappa}{\partial \theta} \Big) \cdot \boldsymbol{g}_\kappa + \Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda \frac{\partial \boldsymbol{q}_\kappa}{\partial \boldsymbol{g}_\kappa} \Big) \cdot \nabla \boldsymbol{g}_\kappa \geq 0. \end{aligned}$$

Note that this inequality is linear in $\dot{\theta}$, $\dot{\boldsymbol{g}}_{\boldsymbol{\kappa}}$, and $\nabla \boldsymbol{g}_\kappa$, and arbitrariness of their values at (X,t) implies that their coefficients must vanish,

$$\frac{\partial \eta_\kappa}{\partial \theta} - \Lambda \frac{\partial \varepsilon_\kappa}{\partial \theta} = 0, \qquad \frac{\partial \eta_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda \frac{\partial \varepsilon_\kappa}{\partial \boldsymbol{g}_\kappa} = 0 \qquad \Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda \frac{\partial \boldsymbol{q}_\kappa}{\partial \boldsymbol{g}_\kappa} \Big)_{\text{sym}} = 0. \tag{15}$$

In the last relation only the symmetric part needs to vanish, because $\nabla \boldsymbol{g}_\kappa = \nabla(\nabla\theta)$ is the symmetric second gradient of the temperature. The remaining part of the inequality (14) becomes

$$\rho_\kappa \sigma = \Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \theta} - \Lambda \frac{\partial \boldsymbol{q}_\kappa}{\partial \theta} \Big) \cdot \boldsymbol{g}_\kappa \geq 0, \tag{16}$$

so that the entropy production $\sigma(\theta, \boldsymbol{g}_\kappa)$ must be non-negative for any values of $(\theta, \boldsymbol{g}_\kappa)$.

The relation (15) and (16) are the general restrictions imposed by the entropy principle. They contain the Lagrange multiplier $\Lambda(\theta, \boldsymbol{g}_\kappa)$ whose physical significance remains to be analyzed in the theory. For further evaluations, more specific constitutive equations of the material model must be considered.

4.1 Transversely isotropic rigid heat conductors

As an example, we shall consider a class of transversely isotropic rigid heat conductors whose material symmetry group is given by

$$\mathcal{G} = \{ G \in \mathcal{O}^+ \mid G\boldsymbol{n} = \boldsymbol{n} \}, \tag{17}$$

which is the group of all rotations about the symmetry axis characterized by the unit vector $\boldsymbol{n}$ in the reference configuration κ. The group $\mathcal{O}^+$ denotes the proper orthogonal group which contains all orthogonal transformations with positive determinant. From the condition (13), the heat flux $\boldsymbol{q}_\kappa = \boldsymbol{q}(\theta, \boldsymbol{g}_\kappa)$ is an invariant vector function with respect to the group $\mathcal{G}$.

Representations of invariant functions with respect to symmetry group of this type have been considered. They are given in the following theorem (the proof is given in [4, 6]):

Theorem. *A function $f = f(A, \boldsymbol{v})$ is an invariant function with respect to the symmetry group $\mathcal{G}$, defined in (17), if and only if it can be represented by $f(A, \boldsymbol{v}) = \widetilde{f}(A, \boldsymbol{v}, \boldsymbol{n})$, where $\widetilde{f}$ is an invariant function with an additional vector variable $\boldsymbol{n}$ with respect to the proper orthogonal group $\mathcal{O}^+$.*

Invariant function with respect to the proper orthogonal group $\mathcal{O}^+$ are also referred to as hemitropic functions.

From the above theorem, the constitutive function for the heat flux $\boldsymbol{q}_\kappa$ and the entropy flux $\boldsymbol{\Phi}_\kappa$ can be represented as hemitropic vector functions of $(\theta, \boldsymbol{g}_\kappa, \boldsymbol{n})$, one scalar and two vector variables. The hemitropic representations can be obtained from the results available in the literature (see [6]).

The constitutive equations of heat flux and entropy flux contain hemitropic vector elements $\{\boldsymbol{g}_\kappa, \boldsymbol{n}, \boldsymbol{n}\times\boldsymbol{g}_\kappa\}$ with coefficients in functions of scalar invariants $\{\theta, \boldsymbol{n}\cdot\boldsymbol{g}_\kappa, \boldsymbol{g}_\kappa\cdot\boldsymbol{g}_\kappa\}$, where $\times$ denotes the vector product. For simplicity, we shall consider the special case that the constitutive functions contain up to linear terms in the temperature gradient $\boldsymbol{g}_\kappa$,

$$\begin{aligned} \boldsymbol{q}_\kappa &= a_0\, \boldsymbol{g}_\kappa + (b_0 + b_1\, (\boldsymbol{n}\cdot\boldsymbol{g}_\kappa))\, \boldsymbol{n} + c_0\, \boldsymbol{n}\times\boldsymbol{g}_\kappa, \\ \boldsymbol{\Phi}_\kappa &= \alpha_0\, \boldsymbol{g}_\kappa + (\beta_0 + \beta_1\, (\boldsymbol{n}\cdot\boldsymbol{g}_\kappa))\, \boldsymbol{n} + \gamma_0\, \boldsymbol{n}\times\boldsymbol{g}_\kappa, \end{aligned} \tag{18}$$

where all the material coefficients are functions of temperature θ only.

From these representations, the relation $(15)_3$ gives

$$\Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda\, \frac{\partial \boldsymbol{q}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big)_{\rm sym} = (\alpha_0 - \Lambda\, a_0)\, I + (\beta_1 - \Lambda\, b_1)\, \boldsymbol{n}\otimes \boldsymbol{n} = 0,$$

where we have noted that

$$\Big(\frac{\partial\, \boldsymbol{n}\times\boldsymbol{g}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big)_{\rm sym} = 0.$$

Since the identity tensor I and the tensor product $\boldsymbol{n}\otimes\boldsymbol{n}$ are functionally independent, it implies that

$$\alpha_0 = \Lambda\, a_0, \qquad \beta_1 = \Lambda\, b_1, \tag{19}$$

from which it follows that Λ must be independent of $\boldsymbol{g}_\kappa$, since the material coefficients are functions of temperature θ only, i.e., $\Lambda = \Lambda(\theta)$.

Let us define a thermodynamic function,

$$\psi_\kappa(\theta, \boldsymbol{g}_\kappa) = \eta_\kappa(\theta, \boldsymbol{g}_\kappa) - \Lambda(\theta)\, \varepsilon_\kappa(\theta, \boldsymbol{g}_\kappa),$$

then from $(15)_2$, we have

$$\frac{\partial \psi_\kappa}{\partial \boldsymbol{g}_\kappa} = \frac{\partial \eta_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda\, \frac{\partial \varepsilon_\kappa}{\partial \boldsymbol{g}_\kappa} = 0,$$

which implies that $\psi_\kappa = \psi_\kappa(\theta)$, and from $(15)_1$, we have

$$\frac{\partial \psi_\kappa}{\partial \theta} = \frac{\partial \eta_\kappa}{\partial \theta} - \Lambda\, \frac{\partial \varepsilon_\kappa}{\partial \theta} - \varepsilon_\kappa\, \frac{\partial \Lambda}{\partial \theta} = -\varepsilon_\kappa\, \frac{\partial \Lambda}{\partial \theta}.$$

Since both ψ_κ and Λ are functions of temperature θ only. it follows that

$$\varepsilon_\kappa = \varepsilon_\kappa(\theta), \qquad \eta_\kappa = \eta_\kappa(\theta),$$

and from the relation $(15)_1$ and the classical thermostatics, we can identify the Lagrange multiplier as the reciprocal of the absolute temperature,

$$\Lambda(\theta) = \frac{1}{\theta}. \tag{20}$$

Note that the function $\psi_\kappa(\theta)$ is a thermodynamic potential,

$$\psi_\kappa = \eta_\kappa - \frac{1}{\theta}\,\varepsilon_\kappa, \qquad \varepsilon_\kappa = \theta^2\,\frac{\partial \psi_\kappa}{\partial \theta}, \qquad \eta_\kappa = \psi_\kappa + \theta\,\frac{\partial \psi_\kappa}{\partial \theta}. \tag{21}$$

Moreover, from (18) and (19), we obtain the relation between the entropy flux and the heat flux,

$$\boldsymbol{\Phi}_\kappa = \frac{1}{\theta}\,\boldsymbol{q}_\kappa + k_0\,\boldsymbol{n} + k_1\,\boldsymbol{n}{\times}\boldsymbol{g}_\kappa, \tag{22}$$

where $k_0 = \beta_0 - b_0/\theta$ and $k_1 = \gamma_0 - c_0/\theta$.

Further results may follow from the entropy production inequality (16), which from (22) becomes

$$\sigma = -\frac{1}{\theta^2}\,(a_0\,\boldsymbol{g}_\kappa{\cdot}\boldsymbol{g}_\kappa + b_0\,\boldsymbol{n}{\cdot}\boldsymbol{g}_\kappa + b_1\,(\boldsymbol{n}{\cdot}\boldsymbol{g}_\kappa)^2) + k_0'\,\boldsymbol{n}{\cdot}\boldsymbol{g}_\kappa \geq 0.$$

The function $\sigma(\theta, \boldsymbol{g}_\kappa)$ attains its minimum, namely 0, at $\boldsymbol{g}_\kappa = 0$, therefore by assuming smoothness, it is necessary that

$$\frac{\partial \sigma}{\partial \boldsymbol{g}_\kappa}\bigg|_{\boldsymbol{g}_\kappa=0} = -\frac{1}{\theta^2}\,b_0\,\boldsymbol{n} + k_0'\,\boldsymbol{n} = 0,$$

which leads to

$$k_0' = \frac{b_0}{\theta^2}. \tag{23}$$

Moreover, the necessary condition for minimum also requires that the second gradient be positive semi-definite,

$$\frac{\partial^2 \sigma}{\partial \boldsymbol{g}_\kappa \partial \boldsymbol{g}_\kappa}\bigg|_{\boldsymbol{g}_\kappa=0} = -\frac{1}{\theta^2}\,(a_0\,I + b_1\,\boldsymbol{n}{\otimes}\boldsymbol{n}) \geq 0,$$

from which it follows

$$a_0 \leq 0, \qquad a_0 + b_1 \leq 0. \tag{24}$$

In summary, the relations (20) through (24) are the general results from the entropy principles for this class of rigid heat conductors. In particular, we shall emphasize the relation between the entropy flux and the heat flux, from $(18)_1$ and (22),

$$\begin{aligned} \boldsymbol{q}_\kappa &= a_0\,\boldsymbol{g}_\kappa + (b_0 + b_1\,(\boldsymbol{n}{\cdot}\boldsymbol{g}_\kappa))\,\boldsymbol{n} + c_0\,\boldsymbol{n}{\times}\boldsymbol{g}_\kappa, \\ \boldsymbol{\Phi}_\kappa &= \frac{1}{\theta}\,\boldsymbol{q}_\kappa + \underline{k_0\,\boldsymbol{n} + k_1\,\boldsymbol{n}{\times}\boldsymbol{g}_\kappa}. \end{aligned}$$

The underlined terms shows that the classical assumption that the entropy flux be defined as the heat flux by the absolute temperature is not valid in general [5]. This fact has long been recognized from kinetic theory of gases and theories of mixtures and porous media. However, few other examples are available from theories as simple as the present one and the case of elastic material bodies [7, 8].

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