Vanishing Relaxation Limit of Viscoelasticity

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Abstract

A rate-type viscoelastic material with very small relaxation time parameter should be almost an elastic material intuitively. In this paper we shall prove that smooth solutions of initial-boundary value problems with periodic boundary conditions of a one-dimensional elastic body can be obtained as the vanishing relaxation limits of the corresponding viscoelastic problems.

1 Introduction

We shall consider the behavior of a one-dimensional solid body modeled as an elastic material as well as a rate-type viscoelastic material in the following problems.

1) An elastic problem:

\[
\frac{\partial v}{\partial t} - \frac{\partial \sigma(\varepsilon)}{\partial x} = 0, \quad \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0,
\]

with the initial and periodic boundary conditions:

\[
(v, \varepsilon)(0, x) = (v_0, \varepsilon_0)(x), \quad x \in [0, 1],
\]
\[
(v, \varepsilon)(t, 0) = (v, \varepsilon)(t, 1), \quad t \geq 0,
\]

where \(v\) is the particle velocity, \(\varepsilon\) is the strain (the spatial derivative of the displacement) and \(\sigma = \sigma_e(\varepsilon)\) is the stress-strain relation.

2) A viscoelastic problem:

\[
\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0, \quad \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0,
\]
\[
\frac{\partial \sigma}{\partial t} - E \frac{\partial v}{\partial x} = \frac{1}{\tau} (\sigma_e(\varepsilon) - \sigma),
\]

with the initial condition and the periodic boundary conditions

\[
(v, \varepsilon, \sigma)(0, x) = (v_0, \varepsilon_0, \sigma_0)(x), \quad x \in [0, 1],
\]
\[
(v, \varepsilon, \sigma)(t, 0) = (v, \varepsilon, \sigma)(t, 1), \quad t \geq 0.
\]

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The viscoelastic model (1.3) was proposed in [1].

Intuitively, one would expect that a material with very small relaxation parameter \( \tau \) is almost an elastic material, and hence consider the solution of the viscoelastic problem as an approximation to the solution of the elastic problem in the limit when \( \tau \) tends to zero. In [2], Riemann problems for the systems (1.1) and (1.3) have been investigated numerically in order to qualify the above statement of approximation in the vanishing relaxation limit. In this paper, we shall consider initial-boundary value problems with periodic boundary conditions, and shall prove that if the elastic problem (1.1), (1.2) has a smooth solution, then it can be obtained as the vanishing relaxation limit of solutions for the corresponding viscoelastic problem (1.3), (1.4).

The relaxation phenomena are usually present, however small, in the behavior of most real material bodies, and hence is often incorporated into the formulation of material models, usually in the form of extra balance laws (see [1, 3], and more generally in the theories of extended thermodynamics [4]). It has been an appealing topic to study conservation laws with relaxation and much work has been done lately, particularly in the behaviors of the vanishing relaxation limit [5, 6, 7, 8, 9]. An obvious advantage from mathematical point of view is that by introducing the fluxes as additional independent variables, some nonlinear systems of conservation laws become hyperbolic semilinear systems. On the other hand through a Chapman-Enskog type expansions, the vanishing relaxation approximations also render dissipative mechanisms and hence give rise to certain stability criteria similar to the more usual zero viscosity limit of the conservation laws. In the present paper, we do not employ a Chapman-Enskog type expansion and instead a Hilbert type expansion in the kinetic theory is used [10].

2 The Main Results

For convenience, we shall rewrite the system (1.3) in the following form

\[
\frac{\partial \mathbf{u}}{\partial t} + F \frac{\partial \mathbf{u}}{\partial x} = \frac{1}{\tau} G(\mathbf{u}),
\]

with the initial and boundary conditions:

\[
\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad x \in [0, 1],
\]

\[
\mathbf{u}(t, 0) = \mathbf{u}(t, 1), \quad t \geq 0,
\]

where \( \mathbf{u} = (v, \varepsilon, \sigma)^T \) and

\[
F = \begin{bmatrix}
0 & 0 & -1 \\
-1 & 0 & 0 \\
-E & 0 & 0
\end{bmatrix}, \quad G(\mathbf{u}) = \begin{bmatrix}
0 \\
0 \\
\sigma_\varepsilon(\varepsilon) - \sigma
\end{bmatrix}.
\]

We wish to show that system (2.1), (2.2) has a unique solution \( \mathbf{u}^\tau \) in some suitable class, for \( \tau \) sufficiently small, and to find the approximate behavior of \( \mathbf{u}^\tau \) as \( \tau \) tends to zero. The equation (2.1) is a system of hyperbolic conservation laws with a stiff coefficient, the small relaxation time parameter. We expect that the set of vectors \( \mathbf{g} \) such that

\[
G(\mathbf{g}) = 0
\]
will play an important role. Obviously from (2.3), a vector \( \mathbf{g} = (g_1, g_2, g_3) \) satisfies (2.4) if and only if
\[
\sigma_e(g_2) = g_3.
\]
We shall call such a vector \( \mathbf{g} \) an equilibrium state of the viscoelastic equation.

In this paper, we shall prove that if the elastic problem (1.1), (1.2) has a smooth solution, then it can be obtained as the vanishing relaxation limit of solutions for the corresponding viscoelastic problem (2.1), (2.2). Therefore, it will be assumed in the sequel that the initial distribution \( \mathbf{u}_0(x) \) in (1.2) is given so that the quasi-linear system (1.1) with the initial and boundary conditions (1.2) admits smooth solutions. The equilibrium state of the viscoelastic problem associated the smooth solution is given by
\[
\begin{align*}
g_1 &= v, \\
g_2 &= \varepsilon, \\
g_3 &= \sigma_e(\varepsilon).
\end{align*}
\]
By a smooth function we mean a differentiable function with bounded derivatives up to a certain order.

More specifically, in this paper, we shall assume that \( \sigma_e \) is a function which can be expressed as a polynomial of \( \varepsilon \) up to the third degree, i.e.,
\[
\sigma_e(\varepsilon) = a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3,
\]
for certain constants \( a_1, a_2, a_3 \), and furthermore the function \( \sigma_e \) satisfies the following condition
\[
0 < \sigma_e'(\varepsilon) < E,
\]
where the positive constant \( E \) is the one in (1.3).

The main result is stated in the following theorem:

**Theorem.** Let \( \mathbf{u}_0(x) \) in (2.2) be a smooth periodic function from \([0, 1]\) to \( \mathbb{R}^3 \). Assume that the elastic problem (1.1), (1.2) has a smooth solution in some fixed time interval \([0, T]\) with \( T < \infty \). Then there exists a \( \tau_0 > 0 \) such that for each \( \tau \) with \( 0 < \tau \leq \tau_0 \) the viscoelastic problem (2.1), (2.2) has a unique solution \( \mathbf{u}^\tau \) such that
\[
\mathbf{u}^\tau \in L^\infty([0, T]; L^2).
\]
Let \( \mathbf{g} = \mathbf{g}(t, x) \) denote the equilibrium state associated with the smooth solution \((v, \varepsilon)\). Then there exists a constant \( C < \infty \), and a \( \delta \geq 0 \) sufficiently small, depending only on the initial data \( \mathbf{u}_0 \), such that
\[
\sup_{\delta \leq t \leq T} \| \mathbf{u}^\tau(t) - \mathbf{g}(t) \|_{L^2} \leq \tau C
\]
Moreover, if the initial data satisfy \( \sigma_0(x) = \sigma_e(\varepsilon_0(x)) \), then (2.9) holds for \( \delta = 0 \).

We shall borrow the idea of [10] to prove the theorem. The basic idea is to use both Hilbert expansion and initial layer expansion to construct a solution of the viscoelastic equation, then to study its behavior when the relaxation time \( \tau \) tends to zero. The proof will be given in the following steps:
2.1 The Hilbert Expansion

We seek an expansion for \( \tilde{u}^\tau \), a solution for (2.1), in the form
\[
\tilde{u}^\tau = g + \tau h_1 + \tau^2 h_2 + \tau^3 h_3 + \cdots
\] (2.10)
where \( g \) is the equilibrium state associated with the solution of the elastic problem, and hence
\[
g(0, x) = (v_0, \varepsilon_0, \sigma_e(\varepsilon_0))(x).
\] (2.11)

We shall disregard the initial and boundary conditions (2.2) for the moment. Inserting (2.10) into (2.1) and collecting terms with equal powers of \( \tau \), we obtain the following relations:
\[
\begin{align*}
\frac{\partial g_1}{\partial t} - \frac{\partial g_3}{\partial x} &= 0, \\
\frac{\partial h_{i1}}{\partial t} - \frac{\partial h_{i3}}{\partial x} &= 0, \\
\frac{\partial h_{i2}}{\partial t} - \frac{\partial h_{i1}}{\partial x} &= 0,
\end{align*}
\] (2.12)
for \( i = 1, 2, 3, \ldots \), and
\[
\begin{align*}
\frac{\partial g_3}{\partial t} - E\frac{\partial g_1}{\partial x} + \sum_{i=1}^{\infty} \tau^i \frac{\partial h_{i3}}{\partial x} - E\sum_{i=1}^{\infty} \tau^i \frac{\partial h_{i1}}{\partial x} &= 0, \\
&= \frac{1}{\tau} \left( \sigma_e(g_2 + \sum_{i=1}^{\infty} \tau^i h_{i2}) - (g_3 + \sum_{i=1}^{\infty} \tau^i h_{i3}) \right).
\end{align*}
\] (2.13)

It is clear that (2.12)_{1,2} are satisfied by the equilibrium state function \( g \). From (2.12)_{3,4}, we have, for \( i = 1 \),
\[
\begin{align*}
\frac{\partial h_{11}}{\partial t} - \frac{\partial h_{13}}{\partial x} &= 0, \\
\frac{\partial h_{12}}{\partial t} - \frac{\partial h_{11}}{\partial x} &= 0.
\end{align*}
\] (2.14)

On the other hand, by the use of the expression (2.6), the equation (2.13) for \( i = 1 \) yields
\[
h_{13} = \sigma_e'(g_2)h_{12},
\] (2.15)
from the zeroth order terms in \( \tau \). Upon insertion of \( h_{13} \) into (2.14), we obtain two equations for the two functions \( h_{11} \) and \( h_{12} \):
\[
\begin{align*}
\frac{\partial h_{11}}{\partial t} - \sigma_e'(g_2)\frac{\partial h_{12}}{\partial x} &= \phi(t, x)h_{12}, \\
\frac{\partial h_{12}}{\partial t} - \frac{\partial h_{11}}{\partial x} &= 0,
\end{align*}
\] (2.16)
which is a linear hyperbolic system by (2.7) and where
\[
\phi(t, x) = \sigma_e''(g_2)\frac{\partial g_2}{\partial x}.
\]

For \( i = 2 \) and \( i = 3 \), following the same procedures we can obtain similar linear hyperbolic systems for \( h_{i1} \) and \( h_{i2} \):
\[
\begin{align*}
\frac{\partial h_{i1}}{\partial t} - \sigma_e'(g_2)\frac{\partial h_{i2}}{\partial x} &= \phi(t, x)h_{i2} + \psi_i(t, x), \\
\frac{\partial h_{i2}}{\partial t} - \frac{\partial h_{i1}}{\partial x} &= 0,
\end{align*}
\] (2.17)
and the relations,
\[
\begin{align*}
    h_{23} &= \sigma'(g_2)h_{22} + \frac{\sigma''(g_2)}{2} h_{12}^2 - \left( \frac{\partial h_{13}}{\partial t} - E \frac{\partial h_{11}}{\partial x} \right), \\
    h_{33} &= \sigma'(g_2)h_{32} + \sigma''(g_2) h_{11} h_{22} + a_3 h_{12}^3 - \left( \frac{\partial h_{23}}{\partial t} - E \frac{\partial h_{21}}{\partial x} \right),
\end{align*}
\]
(2.18)

where \( \psi_i \) are known smooth functions depending on \( x \) and \( t \) only. Note that the system (2.16) can be regarded as a special case of (2.17) for \( i = 1 \) and \( \psi_1 = 0 \).

By applying the results from [11], we know that there exist unique solutions for the linear hyperbolic systems (2.17) with smooth initial data \( h_{ik}(0, x) \) and periodic boundary conditions,
\[
h_{ik}(t, 0) = h_{ik}(t, 1), \quad t \geq 0,
\]
(2.19)
for \( i = 1, 2, 3 \), and \( k = 1, 2 \). Moreover, we shall choose the initial data \( h_{ik}(0, x) \) in such a way that both the initial data \( h_{ik}(0, x) \) and their derivatives \( h'_{ik}(0, x) \) are periodic functions in \([0, 1]\). In this manner \( h_{i1} \) and \( h_{i2} \) can be obtained, and from (2.15), (2.18), so do the functions \( h_{i3} \). We have thus completely determined the functions \( h_1, h_2, \) and \( h_3 \).

### 2.2 The Initial Layer Expansion

We introduce a stretched time variable,
\[
\theta = \frac{t}{\tau},
\]
and consider the function \( u^\tau \) as a function of \( \theta \) and \( x \). The equation (2.1) and the conditions (2.2) become
\[
\frac{1}{\tau} \frac{\partial u^\tau}{\partial \theta} + F \frac{\partial u^\tau}{\partial x} = \frac{1}{\tau} G(u^\tau),
\]
(2.20)
and
\[
\begin{align*}
    u^\tau(0, x) &= u_0(x), \quad x \in [0, 1], \\
    u^\tau(\theta, 0) &= u^\tau(\theta, 1), \quad \theta \geq 0.
\end{align*}
\]
(2.21)

Let the solution of (2.20) and (2.21) be written as
\[
u^\tau(\theta, x) = \tilde{u}^\tau(\tau\theta, x) + I(\theta, x),
\]
(2.22)
where
\[
\tilde{u}^\tau(\tau\theta, x) = g(\tau\theta, x) + \tau h_1(\tau\theta, x) + \tau^2 h_2(\tau\theta, x) + \tau^3 h_3(\tau\theta, x) + \cdots
\]
is already determined to within the initial conditions (2.2) by the Hilbert expansion (2.10). The remaining part of the expression (2.22) is called the initial layer and we also look for an expansion in the form,
\[
I(\theta, x) = I_0(\theta, x) + \tau I_1(\theta, x) + \tau^2 I_2(\theta, x) + \tau^3 I_3(\theta, x) + \cdots
\]
(2.23)
Now inserting \((2.23)\) into \((2.20)\) and collecting the terms of the same order in \(\tau\), we obtain the following equations for the terms of the initial layer:

\[
\begin{align*}
\frac{\partial I_{01}}{\partial \theta} &= 0, \quad \frac{\partial I_{02}}{\partial \theta} = 0, \\
\frac{\partial I_{03}}{\partial \theta} &= \sigma_e(I_{02}) + 3a_3(g_2(I_{02}))^2 + g_2^2I_{02}) - I_{03}, \quad (2.24)
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial I_{11}}{\partial \theta} &= \frac{\partial I_{03}}{\partial x}, \quad \frac{\partial I_{12}}{\partial \theta} = \frac{\partial I_{01}}{\partial x}, \\
\frac{\partial I_{13}}{\partial \theta} &= E\frac{\partial I_{01}}{\partial x} + \sigma_e'(g_2) + 3a_3(I_{02})^2 + (6g_2 + 2I_{02})(h_{12} + I_{12}) - h_{13} - I_{13}, \quad (2.25)
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial I_{21}}{\partial \theta} &= \frac{\partial I_{13}}{\partial x}, \quad \frac{\partial I_{22}}{\partial \theta} = \frac{\partial I_{11}}{\partial x}, \\
\frac{\partial I_{23}}{\partial \theta} &= -I_{23} + \sigma_e'(g_2)(h_{22} + I_{22} - h_{12} - I_{12}) + (3a_3g_2 + a_2)(h_{12} + I_{12})^2 + h_{13} + I_{13} - E\left(\frac{\partial I_{22}}{\partial \tau} + \frac{\partial I_{12}}{\partial \tau}\right), \quad (2.26)
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial I_{31}}{\partial \theta} &= \frac{\partial I_{23}}{\partial x}, \quad \frac{\partial I_{32}}{\partial \theta} = \frac{\partial I_{21}}{\partial x}, \\
\frac{\partial I_{33}}{\partial \theta} &= -\frac{\partial h_{23}}{\partial \tau} + E\left(\frac{\partial I_{21}}{\partial \tau} + \frac{\partial h_{22}}{\partial \tau}\right) + \sigma_e'(g_2)(h_{32} + I_{32}) + (6a_3g_2 + 2a_2)(h_{12}h_{22} + I_{12}I_{22} + h_{12}I_{22} + I_{12}h_{22}) + a_3(h_{12}^3 + I_{12}^3) + 3h_{12}I_{12}(h_{12} + I_{12}) - h_{33} - I_{33}, \quad (2.27)
\end{align*}
\]

and so on.

In view of the initial and boundary conditions \((2.21)\) and the condition \((2.11)\), we shall impose the following initial conditions for \(I_k\),

\[
\begin{align*}
I_{01}(0, x) &= I_{02}(0, x) = 0, \quad I_{03}(0, x) = \sigma_0 - \sigma_e(\varepsilon_0), \\
h_k(0, x) + I_k(0, x) &= 0, \quad k = 1, 2, 3, \ldots, \quad (2.28)
\end{align*}
\]

and the periodic boundary conditions,

\[
I_k(t, 0) = I_k(t, 1).
\]

Note that the equations \((2.24), (2.25), (2.26), (2.27)\) are kind of ordinary differential equations. We shall see in the process of solving these equations that we can have periodic solutions by utilizing the periodicity of the functions \(h_k(t, x)\) in \((2.19)\).

First from \((2.24)\) and \((2.28)\), we can easily obtain

\[
\begin{align*}
I_{01}(\theta, x) &= I_{02}(\theta, x) = 0, \\
I_{03}(\theta, x) &= (\sigma_0 - \sigma_e(\varepsilon_0))e^{-\theta}, \quad (2.29)
\end{align*}
\]

for \(\theta > 0\) and \(x \in [0, 1]\). Similarly, we can treat \((2.25)\) as three ordinary differential
I define $h_{11}(0,x) = (\sigma'_e \varepsilon'_0 - \sigma'_0) e^{-\theta}$

$$I_{12}(\theta, x) = -h_{12}(0,x)$$

$$I_{13}(\theta, x) = -h_{13}(0,x)e^{-\theta} + \int_0^\theta \sigma'_e(g_2)(h_{12} + I_{12})e^{-\theta+\tau} ds,$$  

Similarly, one can also solve (2.26) and (2.27) with condition (2.28) for $I_2(\theta, x)$ and $I_3(\theta, x)$ respectively.

**Remark.** As we have mentioned at the end of the last subsection, that we can impose some restrictions on the initial data $h_k(0,x)$ as long as they satisfy (2.19). In particular, in obtaining (2.30), we have already required that

$$h_{11}(0,x) = \sigma'_0 - \sigma'_e \varepsilon'_0,$$

which is a periodic function for $x \in [0,1]$. We can also impose conditions such that $I_k(\theta, x)$, for $k = 0, 1, 2, 3$, are bounded and periodic functions as $\theta \to \infty$.

### 2.3 The error equation

So far we have constructed the functions: $g$, $h_1$, $h_2$, $h_3$, $I_0$, $I_1$, $I_2$, and $I_3$. Now we define $H^\tau(t,x)$ by

$$H^\tau(t,x) = h_1(t,x) + \tau h_2(t,x) + \tau^2 h_3(t,x) + I_1(\frac{t}{\tau}, x) + \tau I_2(\frac{t}{\tau}, x) + \tau^2 I_3(\frac{t}{\tau}, x),$$

and set

$$u^\tau(t,x) = g(t,x) + I_0(\frac{t}{\tau}, x) + \tau H^\tau(t,x) + \tau^2 \tilde{\omega}^\tau(t,x).$$

This defines $\tilde{\omega}^\tau(t,x)$ as the remaining error terms in the solution $u^\tau$ of (2.1) and (2.2).

In order to estimate the error $\tilde{\omega}^\tau$, we shall insert (2.31) into (2.1). First, let us expand the function $G(u^\tau)$. By (2.3), we know that we only need to calculate $\sigma_e(u^\tau) - u^\tau_3$, because the first two components of $G$ are zeros. We have

$$\frac{1}{\tau}(\sigma_e(u^\tau_2) - u^\tau_3) = \tau^{-1}[\sigma_e(g_2 + I_{02} + \tau H_2^\tau + \tau^2 \tilde{\omega}_2^\tau(t,x)) - (g_3 + I_{03} + \tau H_3^\tau + \tau^2 \tilde{\omega}_3^\tau(t,x))]$$

$$= -\tau^{-1}I_{03} + \sigma'_e(g_2)H_2^\tau - H_3^\tau + \tau(\sigma'_e(g_2)\tilde{\omega}_2^\tau - \tilde{\omega}_3^\tau)$$

$$+ \tau(3a_3 H_2^\tau g_2 + a_2 H_2^\tau) + \tau^2(a_3 H_2^\tau + 6g_3a_3 H_3^\tau \tilde{\omega}_2^\tau + 2a_2 H_2^\tau \tilde{\omega}_2^\tau)$$

$$+ \tau^3(3a_3 H_2^\tau \tilde{\omega}_2^\tau + a_2 \tilde{\omega}_2^\tau) + \tau^4(3a_3 H_2^\tau \tilde{\omega}_2^\tau) + \tau^5(a_3 \tilde{\omega}_2^\tau).$$

Substituting this relation into the right hand side of (2.2), and by the use of the functions $g$, $I_0$, $H^\tau$, we obtain, after some rearrangement, the following error equation for $\tilde{\omega}^\tau(t,x)$:

$$\frac{\partial \tilde{\omega}^\tau}{\partial t} + F \frac{\partial \tilde{\omega}^\tau}{\partial x} = \frac{1}{\tau}\tilde{L}\tilde{\omega}^\tau + \tilde{M}^\tau \tilde{\omega}^\tau + \tau \tilde{Z}(\tilde{\omega}^\tau) + \tau \tilde{A}^\tau,$$  

where

$$\tilde{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma' & -1 \end{bmatrix}, \quad \tilde{M}^\tau = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{Z}_{32}(\tilde{\omega}^\tau) & 0 \end{bmatrix}.$$
and
\[ \tilde{Z}_{32}(\tilde{\omega}^\tau) = (3a_3g_2 + a_2 + 3\tau a_3 H^\tau_2)\tilde{\omega}^\tau_2 + a_3\tau^2\tilde{\omega}^\tau_2. \]

The matrix \( \tilde{M}^\tau = \tilde{M}^\tau(t, x) \) and the vector \( \tilde{A}^\tau = \tilde{A}^\tau(t, x) \) are independent of \( \tilde{\omega}^\tau \), and their entries are bounded periodic functions on \([0, 1], t \in [0, T] \).

Since we have
\[ \tau^2\tilde{\omega}^\tau(t, x) = u^\tau(t, x) - g(t, x) - I_0(\frac{t}{\tau}, x) - \tau H^\tau(t, x), \]
from (2.31), by (2.11) and (2.28), we obtain
\[ \tilde{\omega}^\tau(0, x) = \frac{1}{\tau^2}(u^\tau(0, x) - g(0, x) - I_0(0, x)) = 0. \quad (2.33) \]

To prove the existence of solutions to the equation (2.32) with the initial condition \( \tilde{\omega}^\tau(0, x) = 0 \), we shall first make some transformation to symmetrize the matrix \( F \) of the equation (2.32) in order to apply the results from [12]. Let us define the matrix \( P \) by
\[
P = \begin{bmatrix}
    (\frac{E}{2} - \frac{\sigma'_1}{2})^{\frac{1}{2}} & 0 & (\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}} \\
    0 & -E(\frac{\sigma'_1}{E})^{\frac{1}{2}} & (\frac{\sigma'_1}{E})^{\frac{1}{2}} \\
    -(\frac{E}{2} - \frac{\sigma'_2}{2})^{\frac{1}{2}} & 0 & (\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}}
\end{bmatrix}.
\]

By (2.7), it is invertible and
\[
P^{-1} = \begin{bmatrix}
    \frac{1}{2}(\frac{E}{2} - \frac{\sigma'_1}{2})^{\frac{1}{2}} & 0 & -\frac{1}{2}(\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}} \\
    \frac{1}{2E}(\frac{1}{2} - \frac{\sigma'_1}{2E})^{\frac{1}{2}} & -\frac{1}{2}(\frac{\sigma'_1}{E})^{\frac{1}{2}} & \frac{1}{2E}(\frac{1}{2} - \frac{\sigma'_1}{2E})^{\frac{1}{2}} \\
    \frac{1}{2}(\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}} & 0 & \frac{1}{2}(\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}}
\end{bmatrix}.
\]

Let \( \omega^\tau = P\tilde{\omega}^\tau \) and multiplied by the matrix \( P \), the equation (2.32) becomes
\[ \frac{\partial \omega^\tau}{\partial t} + V \frac{\partial \omega^\tau}{\partial x} = \frac{1}{\tau}L\omega^\tau + M^\tau\omega^\tau + \tau Z(\omega^\tau)\omega^\tau + \tau A^\tau, \quad (2.34) \]
where
\[ V = \begin{bmatrix}
    -\sqrt{E} & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & \sqrt{E}
\end{bmatrix} \]
and
\[
L = \begin{bmatrix}
    (\frac{1}{2} - \frac{\sigma'_1}{2E}) & (\frac{\sigma'_1}{E})^{\frac{1}{2}}(\frac{1}{2} - \frac{\sigma'_1}{2E})^{\frac{1}{2}} & (\frac{1}{2} - \frac{\sigma'_2}{2E}) \\
    (\frac{\sigma'_1}{E})^{\frac{1}{2}}(\frac{1}{2} - \frac{\sigma'_1}{2E})^{\frac{1}{2}} & \frac{\sigma'_1}{E} & (\frac{\sigma'_1}{E})^{\frac{1}{2}}(\frac{1}{2} - \frac{\sigma'_2}{2E})^{\frac{1}{2}} \\
    (\frac{1}{2} - \frac{\sigma'_2}{2E}) & (\frac{\sigma'_1}{E})^{\frac{1}{2}}(\frac{1}{2} - \frac{\sigma'_1}{2E})^{\frac{1}{2}} & (\frac{1}{2} - \frac{\sigma'_2}{2E})
\end{bmatrix}.
\]
are symmetric matrices, and

\[
Z(\omega^\tau) = -Z_{32}(\omega^\tau) = \begin{bmatrix}
\frac{1}{2E} & -\frac{1}{E} (\frac{\sigma'}{E})^{-\frac{1}{2}} (\frac{1}{2} - \frac{\sigma'}{2E}) & \frac{1}{2E} \\
\frac{1}{2E} (\frac{\sigma'}{E})^{-\frac{1}{2}} (\frac{1}{2} - \frac{\sigma'}{2E}) & -\frac{\sigma'}{E} & \frac{1}{2E} (\frac{\sigma'}{E})^{-\frac{1}{2}} (\frac{1}{2} - \frac{\sigma'}{2E})^{-\frac{1}{2}} \\
\frac{1}{2E} & -\frac{1}{E} (\frac{\sigma'}{E})^{-\frac{1}{2}} (\frac{1}{2} - \frac{\sigma'}{2E})^{-\frac{1}{2}} & \frac{1}{2E}
\end{bmatrix},
\]

where

\[
Z_{32}(\omega^\tau) = (3a_3g_2 + a_2 + 3\tau a_3 H_2^\tau)
\cdot \left\{ \frac{1}{2E} \left( \frac{1}{2} - \frac{\sigma'}{2E} \right)^{-\frac{1}{2}} \omega_1^\tau - \frac{1}{E} \left( \frac{\sigma'}{E} \right)^{-\frac{1}{2}} \omega_2^\tau + \frac{1}{2E} \left( \frac{1}{2} - \frac{\sigma'}{2E} \right)^{-\frac{1}{2}} \omega_3^\tau \right\}
+ a_3 \tau^2 \left\{ \frac{1}{2E} \left( \frac{1}{2} - \frac{\sigma'}{2E} \right)^{-\frac{1}{2}} \omega_1^\tau - \frac{1}{E} \left( \frac{\sigma'}{E} \right)^{-\frac{1}{2}} \omega_2^\tau + \frac{1}{2E} \left( \frac{1}{2} - \frac{\sigma'}{2E} \right)^{-\frac{1}{2}} \omega_3^\tau \right\}^2.
\]

Also we have denoted \( M^\tau = P \tilde{M}^\tau P^{-1} \) and \( A^\tau = P \tilde{A}^\tau \) in (2.34).

Since \( \tilde{L} \) and \( L \) have the same eigenvalues, and the eigenvalues of \( \tilde{L} \) are \( \lambda_1 = 0 \), \( \lambda_2 = 0 \), \( \lambda_3 = -1 \), non-positive, the symmetric matrix \( L \) is negative semi-definite, i.e., for any vector \( \omega \), we have

\[
(\omega, L\omega) \leq 0.
\]

### 2.4 Completion of the proof

Note that the equation (2.34) is non-linear due to the presence of the term \( Z(\omega^\tau)\omega^\tau \). We need the following lemma, which allows us to handle an iteration scheme on the linearized equation of (2.34).

**Lemma.** Let \( \gamma(t, x) \) be a given function such that

\[
\sup_{0 \leq t \leq T} \|\gamma(t)\|_{L^2} \leq \tau \alpha_1, \quad \sup_{0 \leq t \leq T} \|\gamma(t)\|_{H^1} \leq \alpha_2,
\]

and let \( \omega(t, x) \) satisfy the linear system

\[
\frac{\partial \omega}{\partial t} + V \frac{\partial \omega}{\partial x} = \frac{1}{\tau} L\omega + M^\tau \omega + \tau Z(\gamma)\omega + \tau A^\tau,
\]

and the initial condition,

\[
\omega(0, x) = 0, \quad x \in [0, 1].
\]

Then for any \( \tau \) sufficiently small, we have

\[
\sup_{0 \leq t \leq T} \|\omega(t)\|_{L^2} \leq \tau \alpha_1, \quad \sup_{0 \leq t \leq T} \|\omega(t)\|_{H^1} \leq \alpha_2.
\]

**Proof:** From the theory of linear hyperbolic systems [12] and the results in [10] we know that Eq. (2.36) has a unique solution in \( L^\infty([0, T], H^1) \), and there exists a constant
such that (2.38) holds. Taking the inner product of (2.36) with $\omega$, using (2.35), and integrating over $x$, and using the periodicity of $\omega$, we obtain

$$
\frac{1}{2} \frac{\partial}{\partial t} \| \omega(t) \|_{L^2}^2 \leq c_1 \| \omega(t) \|_{L^2}^2 + \tau c_2 \| \omega(t) \|_{L^2}^2 + \tau \int_0^1 \langle \omega, Z(\gamma) \omega \rangle \, dx,
$$

(2.39)

where $c_1$ and $c_2$ denote the bounds of $M^\tau$ and $A^\tau$, respectively. On the other hand, by using the elementary form of Sobolev's inequality along with (2.38) we have

$$
\| Z(t) \|_{L^\infty} = \sup_{0 \leq x \leq 1} | Z(t, x) |^2 \leq 2 \| Z(t) \|_{L^2} \| Z(t) \|_{H^1} \leq 2 \| Z(t) \|_{H^1}^2,
$$

and hence

$$
\| Z(t) \|_{L^\infty} \leq \sqrt{2} \| Z(t) \|_{H^1} \leq \sqrt{2} \alpha_2.
$$

We use this in the integral term in (2.39) and obtain

$$
\frac{\partial}{\partial t} \| \omega(t) \|_{L^2} \leq c_1 \| \omega(t) \|_{L^2} + \tau c_2 + \tau \sqrt{2} \alpha_2 c_3 \| \omega(t) \|_{L^2},
$$

(2.40)

where $c_3$ is a constant that comes from the bound of $Z$. Now integrating (2.40) from 0 to $t$, we have

$$
\| \omega(t) \|_{L^2} \leq \tau c_2 t + \int_0^t (c_1 + \tau \sqrt{2} \alpha_2 c_3) \| \omega(s) \|_{L^2} \, ds.
$$

Applying Gronwall's inequality we obtain

$$
\| \omega(t) \|_{L^2} \leq \tau c_2 t \exp((c_1 + \tau \sqrt{2} \alpha_2 c_3)t).
$$

So it follows that

$$
\sup_{0 \leq t \leq T} \| \omega(t) \|_{L^2} \leq \tau c_2 T \exp((c_1 + \tau \sqrt{2} \alpha_2 c_3)T).
$$

Let $\alpha_1$ be a constant such that

$$
\alpha_1 \geq c_2 T \exp((c_1 + \tau \sqrt{2} \alpha_2 c_3)T),
$$

then it follows

$$
\sup_{0 \leq t \leq T} \| \omega(t) \|_{L^2} \leq \tau \alpha_1.
$$

We have proved the lemma. □

We shall solve (2.34) by using the above lemma and the principle of contraction mappings. Define the iteration scheme $\omega^\tau_N$ by

$$
\omega^\tau_0 = 0
$$

(2.41)

and

$$
\left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) \omega^\tau_{N+1} = \frac{1}{\tau} L \omega^\tau_{N+1} + M^\tau \omega^\tau_{N+1} + \tau Z(\omega^\tau_N) \omega^\tau_{N+1} + \tau \omega^\tau_N,
$$

$$
\omega^\tau_{N+1}(0, x) = 0, \quad N = 0, 1, 2, \ldots.
$$

(2.42)
In view of the above lemma, the iterates satisfy the estimates
\[
\sup_{0 \leq t \leq T} \| \omega_N^\tau(t) \|_{L^2} \leq \tau \alpha_1 \quad \sup_{0 \leq t \leq T} \| \omega_N^\tau(t) \|_{H^1} \leq \alpha_2
\] (2.43)
which are independent of N. From Eq. (2.42) we have
\[
\left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) (\omega_{N+1}^\tau - \omega_N^\tau) = \frac{1}{\tau} L(\omega_{N+1}^\tau - \omega_N^\tau) + M(\omega_{N+1}^\tau - \omega_N^\tau) + \tau Z(\omega_N^\tau) \omega_{N+1}^\tau - \tau Z(\omega_{N-1}^\tau) \omega_N^\tau
\]
\[
= \frac{1}{\tau} L(\omega_{N+1}^\tau - \omega_N^\tau) + M(\omega_{N+1}^\tau - \omega_N^\tau) + \tau Z(\omega_N^\tau) (\omega_{N+1}^\tau - \omega_N^\tau) + \tau (Z(\omega_N^\tau) - Z(\omega_{N-1}^\tau)) \omega_N^\tau
\]
\[
= \frac{1}{\tau} L(\omega_{N+1}^\tau - \omega_N^\tau) + M(\omega_{N+1}^\tau - \omega_N^\tau) + \tau Z(\omega_N^\tau) (\omega_{N+1}^\tau - \omega_N^\tau) + \tau Z'(\xi) \omega_N^\tau (\omega_N^\tau - \omega_{N-1}^\tau),
\]
where \( \xi = \lambda \omega_N^\tau + (1 - \lambda) \omega_{N-1}^\tau \) for some constant \( \lambda \) between 0 and 1 and \( Z'(\gamma) \) is the derivative with respect to \( \gamma \). Taking the inner product of this equation with \( (\omega_{N+1}^\tau - \omega_N^\tau) \), then integrating over \( t \), we have
\[
\| \omega_{N+1}^\tau(t) - \omega_N^\tau(t) \|_{L^2} \leq \int_0^t (c_1 + \tau c_2) \| \omega_{N+1}^\tau(s) - \omega_N^\tau(s) \|_{L^2} ds + \tau c_3 \int_0^t \| \omega_N^\tau(s) - \omega_{N-1}^\tau(s) \|_{L^2} ds.
\]
Using Gronwall’s inequality, we obtain
\[
\| \omega_{N+1}^\tau(t) - \omega_N^\tau(t) \|_{L^2} \leq \tau c_3 \exp((c_1 + \tau c_2)T) \int_0^t \| \omega_N^\tau(s) - \omega_{N-1}^\tau(s) \|_{L^2} ds
\]
\[
\leq \tau c_3 T \exp((c_1 + \tau c_2)T) \sup_{0 \leq t \leq T} \| \omega_N^\tau(t) - \omega_{N-1}^\tau(t) \|_{L^2}.
\]
Take \( c_4 = c_3 T \exp((c_1 + \tau c_2)T) \), we have
\[
\sup_{0 \leq t \leq T} \| \omega_{N+1}^\tau(t) - \omega_N^\tau(t) \|_{L^2} \leq \tau c_4 \sup_{0 \leq t \leq T} \| \omega_N^\tau(t) - \omega_{N-1}^\tau(t) \|_{L^2}. \quad (2.44)
\]
From this inequality it follows immediately that
\[
\sup_{0 \leq t \leq T} \| \omega_{N+1}^\tau(t) - \omega_N^\tau(t) \|_{L^2} \leq (\tau c_4)^{N+1} \alpha_1, \quad N = 0, 1, 2, \ldots
\]
Consequently, for \( \tau \) sufficiently small, the sequence \( \{ \omega_N^\tau(t) \} \) is a Cauchy sequence in \( L^\infty([0, T], L^2) \). Therefore, the limit exists and we denote it by \( \omega^\tau(t) \), which satisfies the estimate (2.37), i.e., this limit is in \( L^\infty([0, T], L^2) \).

From the Eq. (2.42), we see that we can express
\[
\frac{\partial \omega_N^\tau(t, x)}{\partial t} + V \frac{\partial \omega_N^\tau(t, x)}{\partial x}
\]
in terms of something that converges in \( L^\infty([0, T], L^2) \) as \( N \to \infty \). This limit is denoted by
\[
\Psi^\tau = \frac{1}{\tau} L \omega^\tau + M^\tau \omega^\tau + \tau Z(\omega^\tau) \omega^\tau + \tau A^\tau.
\]
Now let $\phi(t, x)$ be a $C^\infty$ function from $[0, T] \times [0, 1]$ to $\mathbb{R}^3$, periodic in $x$ and of compact support in $t$ and in $x$. We have, as $N \to \infty$,

$$
\int_0^T \int_0^1 \left( \phi(t, x), \frac{\partial \omega_N(t, x)}{\partial t} + \frac{\partial \omega_N(t, x)}{\partial x} \right) dx dt \to \int_0^T \int_0^1 \left( \phi(t, x), \Psi^\tau(t, x) \right) dx dt.
$$

But it is obvious that

$$
\int_0^T \int_0^1 \left( \phi(t, x), \frac{\partial \omega_N(t, x)}{\partial t} + V \frac{\partial \omega_N(t, x)}{\partial x} \right) dx dt
= -\int_0^T \int_0^1 \left( \frac{\partial \phi(t, x)}{\partial t}, \omega_N^\tau(t, x) \right) dx dt - \int_0^T \int_0^1 \left( \frac{\partial \phi(t, x)}{\partial x}, V \omega_N^\tau(t, x) \right) dx dt
\to -\int_0^T \int_0^1 \left( \frac{\partial \phi(t, x)}{\partial t}, \omega^\tau(t, x) \right) dx dt - \int_0^T \int_0^1 \left( \frac{\partial \phi(t, x)}{\partial x}, V \omega^\tau(t, x) \right) dx dt.
$$

Therefore, $\Psi^\tau(t, x)$ is identified with the distribution derivatives in $t$ and in $x$ of $\omega^\tau$, that is,

$$
\Psi^\tau(t, x) = \frac{\partial \omega^\tau(t, x)}{\partial t} + V \frac{\partial \omega^\tau(t, x)}{\partial x},
$$

and it follows that $\omega^\tau$ satisfies the error equation almost everywhere with each term taking values in $L^2([0, T], \mathbb{R}^3)$. Then (2.8) holds from the fact that $\omega^\tau \in L^\infty$. Now let us prove (2.9). From (2.38) and the construction of the solution of (1.3), we know that

$$
\| \omega^\tau(t) \|_{L^2} = \frac{1}{\tau^2} \| u^\tau(t) - g(t) - I_0 \left( \frac{t}{\tau} \right) - \tau H^\tau(t) \|_{L^2} \leq \tau \alpha_1,
$$

therefore,

$$
\| u^\tau(t) - g(t) - I_0 \left( \frac{t}{\tau} \right) - \tau H^\tau(t) \|_{L^2} \leq \alpha_1 \tau^3, \quad t \in [0, T],
$$

so,

$$
\| u^\tau(t) - g(t) - I_0 \left( \frac{t}{\tau} \right) \|_{L^2} \leq \alpha_1 \tau^3 + \beta_1 \tau, \quad t \in [0, T], \quad (2.45)
$$

where $\beta_1$ is the bound of $H^\tau$.

From the construction of $I_0$, we have by (2.29)

$$
I_0(\theta, x) \to 0 \quad \text{as} \quad \theta \to \infty, \quad (2.46)
$$

so for any sufficiently small number $\delta > 0$, and for $t > \delta$, $t < T$, (2.46) is equivalent to the following

$$
I_0 \left( \frac{t}{\tau} \right) \to 0 \quad \text{as} \quad \tau \to 0.
$$

Therefore, from (2.45) we have

$$
\sup_{\delta \leq t \leq T} \| u^\tau(t) - g(t) \|_{L^2} \leq \beta_2 e^{-\frac{\tau}{2}} + \alpha_1 \tau^3 + \beta_1 \tau < C \tau,
$$

where $C$ is a positive real number. Moreover, If we take the initial data $u(0, x)$ satisfy $u_3(0, x) = \sigma_x(u_2(0, x))$, namely, we start with an equilibrium state, then $I_0 = 0$. By (2.45) it is clear that the above relation holds for $\delta = 0$. We have completed the proof of the theorem. □
2.5 Remarks

It is interesting to see the construction of $u^\tau$ from the above proof. As we know,

$$u^\tau(t, x) = g(t, x) + I_0(\frac{t}{\tau}, x) + \tau(h_1(t, x) + I_1(\frac{t}{\tau}, x))$$

$$+ \tau^2\{h_2(t, x) + I_2(\frac{t}{\tau}, x) + \tau h_3(t, x) + \tau I_3(\frac{t}{\tau}, x) + P^{-1}\omega^\tau(t, x)\},$$

so the theorem states the following

$$\sup_{0 \leq t \leq T} \|u^\tau(t) - g(t) - I_0(\frac{t}{\tau}) - \tau h_1(t) - \tau I_1(\frac{t}{\tau})\|_{L^2} \leq \tau^2 C.$$  

Note that the boundary layer behavior of $u$ is included up to order $\tau^2$. Therefore, if we take the approximate functions $\hat{u}^\tau(t, x)$ as

$$\hat{u}^\tau(t, x) = g(t, x) + I_0(\frac{t}{\tau}, x) + \tau h_1(t, x) + \tau I_1(\frac{t}{\tau}, x),$$

or as

$$\hat{u}^\tau(t, x) = g(t, x) + I_0(\frac{t}{\tau}, x) + \tau h_1(t, x) + \tau I_1(\frac{t}{\tau}, x)$$

$$+ \tau^2 h_2(t, x) + \tau^2 I_2(\frac{t}{\tau}, x),$$

then they are equal to $u^\tau(t, x)$ within terms of $O(\tau^2)$ or $O(\tau^3)$ respectively.

Our proof also works for the function $\sigma_\varepsilon$ that can be expressed in a polynomial of $n$th degree ($n$ is an integer) as long as $\sigma_\varepsilon$ satisfies (2.7). Indeed, for

$$\sigma_\varepsilon(\varepsilon) = a_1 \varepsilon + a_2 \varepsilon^2 + \cdots + a_n \varepsilon^n,$$

first, with the Hilbert expansion, we can get equations exactly like (2.12), (2.13), (2.14), (2.15), (2.16), (2.17) and (2.18). Then for the initial layer expansion, we also can get similar equations like (2.24), (2.25), (2.26) and (2.27). The rest of the proof is the same.

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