Continuum Mechanics and Thermodynamics manuscript No.
(will be inserted by the editor)

Further Remarks on Euclidean objectivity and the principle of material frame-indifference

I-Shih Liu

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa postal 68530, Rio de Janeiro, 21945-970, Brasil
(e-mail: liu@im.ufrj.br)

The date of receipt and acceptance will be inserted by the editor

Abstract. Murdoch [1] claimed that purely objective considerations imply the restrictions imposed upon response functions by the principle of invariance under superposed rigid body motions. In response to the criticism raised by Liu [2], he failed to recognize the mathematical implication of the condition of objectivity, as pointed out by Liu, that if a response function for an observer is given, the corresponding one for any other observer can be defined uniquely. Consequently his criticism against Liu's counter-example, based on the assumption of observer agreement concerning response functions of different observers, cannot be taken as a valid argument. Furthermore, regarding "objective considerations" as playing the central role in continuum physics, besides our disagreement, especially on the assumption that it is possible to contemplate arbitrary relative motions between two given observers, it is impractical, because, it cannot be stated unambiguously in mathematical terms, and some physical intuitions need to be purposely conceived for occasional cases. Therefore, it runs against the rational spirit deeply cherished in Modern Continuum Mechanics.

Key words: observer, objectivity, response function, material frame-indifference
PACS: 83.10Ff

1 Introduction

The essential idea of the principle of material frame-indifference (MFI) – *the response of a material is the same for all observers* ([4], Sect. 14) – is simple and universally accepted as a guiding principle in establishing sensible physical material models. Unfortunately, interpretation of this simple idea into mathematical statements has been the subject of lengthy controversy, as recently as the dispute between Murdoch [1] and Liu [2]. This note is a reply to the criticisms of Murdoch [3] on [2] as well as the results of a lengthy exchange of each other's viewpoints.

Murdoch [1] claimed to have proved that objectivity considerations, based on the requirements for observer consensus (codified as O.1-5 postulated in [1]), imply the standard restrictions imposed upon response functions by the principle of invariance under superposed rigid body motions (dubbed "isrbm" in [1] and equivalent to the condition of material objectivity in [2]). In response [3] to the criticism raised by Liu, he dismisses the counter-examples of [2] with a different interpretation of response functions, and claims that it is consistent with all objective considerations O.1-5, and the condition of 'standard restriction' is also satisfied. However, Murdoch remains skeptical about the legitimacy of such a material model for an obvious reason that the response function is frame-dependent.

On the other hand, with the different interpretation described in [3], although the example is consistent with all the objectivity considerations O.1-5, it does not satisfy the *correct* standard restriction to be derived later in Sect. 4.2. Therefore, it remains a counter-example from the viewpoint of Murdoch's interpretation.

The arguments against the counter-examples in the criticism [3] rest mainly on the assumption O.4 in [1], concerning observer consensus *on the nature of any given ideal material.* The criticism is innocuous, not only because the assumption O.4 is vaguely stated and hence does not render a clear mathematical interpretation, but also the assumption is superfluous. Indeed, it has been pointed out in [2] and will be emphasized in Sect. 4 that if the response function for an observer is given, the corresponding one for any other observer can be determined uniquely from the objectivity condition.

In [3], concerning one of the response function considered in [2] (Example 1, Sect. 5), Murdoch used it to amplify the tenet of the assumptions O.4 and O.5, for the existence of some other possibilities of the response function for other observers, other than the one that can be determined from the condition of objectivity from Liu's viewpoint. His arguments were based on the assumption that any observer can have "his/her choice of reference configuration". In Sect. 4.3, we shall see that the assumption is inconsistent with the condition of objectivity.

2 Observer and frame of reference

Liu [2] regards a change of observer as a physical term for a change of frame of reference in (classical) space-time ([4,5]). In [1,3], in addition to the usual change of frame of reference in (neo-classical) space-time ([6]), Murdoch imposes a set of metaphysical statements on a change of observer (notably, the assumptions O.4 and O.5). At least, there seems no disagreement between [1] and [2] that a change of observer entails a change of frame as its basic feature. The use of *classical* instead of *neo-classical* space-time is not central to the present discussions (see Remark 3 below).

For a change of observer from ϕ to ϕ^*, the bijective correspondence between $\boldsymbol{x}$ and $\boldsymbol{x}^*$ is defined as a Euclidean transformation,

$$\boldsymbol{x}^* = Q(t)(\boldsymbol{x} - \boldsymbol{x}_o) + \boldsymbol{c}(t), \qquad t^* = t + a, \tag{1}$$

for *some* $\boldsymbol{x}_o \in \mathcal{E}$, $\boldsymbol{c}(t) \in \mathcal{E}$, $a \in I\!R$, and *some* $Q(t) \in \mathcal{O}$, where $\mathcal{O}$ is the group of orthogonal transformations on the translation space V of the Euclidean space $\mathcal{E}$. In other words, the observer ϕ^* is in a time-dependent relative motion with respect to the observer ϕ, with a relative translation $\boldsymbol{c}(t)$ of the origin $\boldsymbol{x}_o$ and a distance-preserving transformation $Q(t)$.

It is obvious that as long as the observer ϕ^* is designated there is *one and only one* relative motion with respect to the observer ϕ. Consequently, for two different observers ϕ and ϕ^*, that "it is possible to contemplate *any number* of relative motions of ϕ^* with respect to ϕ", as stated in [1,3] for the proof of the assertion that objectivity considerations implies *isrbm*", is beyond our comprehension.[1]

Remark 1. Usually, for different observers, agreements on *measurements* of (experimental) results in length, time and other physical units[2] are of their major concern. Unlike Murdoch's embodiment of the assumptions on ideal material behaviors (O.4 and O.5) into the definition of change of observers, the traditional rendering of postulations concerning material behaviors, in Rational Continuum Mechanics, are formulated as independent hypotheses, such as the principles of determinism, local action, and material frame-indifference [4,5].

Remark 2. In the proof of Murdoch's assertion mentioned before, the transformation $Q(t)$ involved in (1) is shown to be a product of the form $Q_2^T Q_1$. Since we disagree with the proof of this result, concerning the assumption that it is possible to contemplate any number of relative motions between

[1] A proof of the same assertion for the case of elastic materials is given in Sect. 2.3 of [3]. It is based on the same arguments employing two relative motions between two observers.

[2] "The dimensional considerations are rarely presented in the change of observer, because they do not offer any peculiar difficulties in mechanics and may easily be included" (cited from the remark in [4], Sect. 17). Thus, I refrain myself from further commenting on the scaling discussion in Sect. 4 of [3].

two observers, the conclusion followed from this result that $Q(t)$ must be a proper orthogonal transformation is void from our point of view. Consequently, the question of whether different observers should agree on the orientation of the corresponding vector spaces remains a question of choice. As for optically-active liquids, in our opinion, it is a question of whether, in his experimental measurements, an observer can or cannot detect the differences of material behaviors under left-handed or right-handed optically polarized light. In other words, it concerns the symmetry properties of materials (e.g., whether the symmetry group be a subgroup of the special linear group, i.e., the unimodular group with positive determinant), rather than a matter of agreement of orientation between different observers.

Remark 3. If neo-classical space-time is considered, then one should distinguish the Euclidean spaces $\mathcal{E}$ and $\mathcal{E}^*$ relative to different observers ϕ and ϕ^* respectively as noted in [1,3]. However, since the change of observer is an isometry that leaves distances unchanged, characterized by $Q(t) \in L(V, V^*)$ as given in (1), the spaces $\mathcal{E}$ and $\mathcal{E}^*$, and correspondingly, the translation spaces V and V^*, are isomorphic to each other, while the corresponding times t and t^* are related to the same intrinsic instant. Moreover, for any vector $\boldsymbol{v} \in V$, there is one and only one vector $\boldsymbol{v}^* = Q(t)\boldsymbol{v} \in V^*$. Therefore, V^* can be regarded as the vector space $\{Q(t)\boldsymbol{v} : \boldsymbol{v} \in V\}$ isomorphic to V. Accordingly, for any domain $\mathcal{D}$, a subspace in V or its tensor space $V \otimes V$, e.g., $\mathrm{Sym}(V)$, the corresponding domain $\mathcal{D}^*$ in V^* or $V^* \otimes V^*$ is well defined and is isomorphic to $\mathcal{D}$. From these observations, the present formulation of the basic ideas in classical space-time can easily be adapted to a mathematically more elaborated but essentially equivalent one in neo-classical space-time.

3 Galilean invariance and Euclidean objectivity

In this section, we shall emphasize that the objectivity considerations for forces, and hence for the stress tensor, are independent of any postulation on material behavior.

Let s, $\boldsymbol{u}$, and T be scalar-, vector-, (second order) tensor-valued functions respectively. If relative to a change of frame from ϕ to ϕ^* given by the Euclidean transformation (1),

$$s(\phi^*) = s(\phi), \qquad \boldsymbol{u}(\phi^*) = Q(t)\,\boldsymbol{u}(\phi), \qquad T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^T,$$

then s, $\boldsymbol{u}$ and T are called *objective* scalar, vector and tensor quantities, respectively, with respect to Euclidean transformations. Moreover, they are said to be objective with respect to Galilean transformation if the orthogonal tensor $Q(t)$ is constant in time and $\boldsymbol{c}(t)$ is linear in time in the transformation (1). For simplicity, we write $f = f(\phi)$ and $f^* = f(\phi^*)$, the value of f observed in ϕ and ϕ^* respectively.

It is known that the acceleration of a motion is not a Euclidean objective vector quantity but it is Galilean objective. It is also well-known that the equation of motion in inertial frames is required to be Galilean invariant. Therefore, if ϕ is an inertial frame, and ϕ^* is a frame differed from ϕ by a Galilean transformation, then we have

$$\int_V \rho\,\ddot{\boldsymbol{x}}\,dv = \int_V \rho\,\boldsymbol{b}\,dv + \int_{\partial V} \boldsymbol{t}\,da, \qquad \int_{V^*} \rho^*\ddot{\boldsymbol{x}}^* dv = \int_{V^*} \rho^*\boldsymbol{b}^* dv + \int_{\partial V^*} \boldsymbol{t}^* da,$$

where V is a material region, $\boldsymbol{b}$ is the body force, and $\boldsymbol{t}$ is the surface traction in ϕ, while the star-quantities are the corresponding ones in ϕ^*. It is universally assumed that the mass density is a Euclidean objective scalar quantity, $\rho^* = \rho$, and since $\ddot{x}^* = Q(t)\ddot{x}$ is Galilean objective, it follows by comparing the above two relations that the forces, $\boldsymbol{b}^* = Q(t)\boldsymbol{b}$ and $\boldsymbol{t}^* = Q(t)\boldsymbol{t}$, must be, at least, Galilean objective vector quantities.

Furthermore, in order to be able to obtain the equation of motion in an arbitrary frame of reference, it is generally assumed that the forces $\boldsymbol{b}$ and $\boldsymbol{t}$ are, not only Galilean objective, but also Euclidean objective vector quantities. From this perspective, we emphasize that Euclidean objectivity of forces is an accepted hypothesis independent of any material behavior.

Remark 4. It follows from the above discussion that the Cauchy stress tensor is a Euclidean objective tensor quantity irrespective of any postulation on material behavior. Therefore, objectivity considerations are not essential in the formulation of the principle of material frame-indifference. The essential idea that the response of a material is the same for all observers can be stated as *form invariance* of the response function for any *objective*[3] constitutive quantity (see e.g. the principle of MFI as stated in [5,7]) or in the version of invariance under superposed rigid body motions (as a consequence of objectivity and form invariance, referred to as the condition of material objectivity in [2] and the standard restrictions in [1]).[4]

4 Conditions of objectivity and the response function

Although it has been pointed out in [2], we shall emphasize again that once a response function is given for an observer, the corresponding response function for any other observer can be defined through the condition of objectivity uniquely. Consequently, there should be no requirement regarding how the response function be given for any other observer, like O.4 in [1]. For simplicity, only mechanical theory will be discussed. Constitutive theories involving thermodynamic properties, mixtures, porous media, electromagnetic effects, liquid crystals, and materials with micro-structures etc. have mostly been successfully formulated along the same rational manner in the literature.

Let $T(X,t)$ be the value of the Cauchy stress tensor relative to the observer ϕ. According to the principle of determinism [4], we postulate the constitutive equation as the following functional relation in the material description,

$$T(X,t) = \mathcal{F}_\phi(\chi^t, X, t), \qquad X \in \mathcal{B}, \tag{2}$$

where $\chi^t : \mathcal{B} \times I\!R^+ \to \mathcal{E}$ is the past history of the motion, i.e., $\chi^t(Y,s) = \chi(Y, t-s)$ and in particular, $\chi^t(X,0) = \boldsymbol{x} \in \mathcal{E}$ is the current position of the material point X at time t.[5] The dependence on X is to allow inhomogeneity of material body, while the inclusion of the present time t as a variable is intended for later discussions in Sect. 5.

Now, let ϕ^* be an arbitrary observer in relative motion with respect to the observer ϕ by a Euclidean transformation of the form (1), and $T^*(X,t^*)$ be the Cauchy stress in Q^*. Since the Cauchy stress is a Euclidean objective tensor, we have

$$T^*(X,t^*) = Q(t)T(X,t)Q(t)^T, \qquad X \in \mathcal{B}, \tag{3}$$

in which $Q(t)$ is *the* orthogonal transformation associated with the relative motion of *the* observer ϕ^* with respect to the observer ϕ.

From the response function $\mathcal{F}_\phi(\chi^t, X, t)$ for the observer ϕ given in (2), and for the corresponding past history of motion $(\chi^t)^*$ in Q^* given by

$$\chi^*(Y, t^*-s) = (\chi^t)^*(Y,s) = Q(t-s)\big(\chi^t(Y,s) - \boldsymbol{x}_o\big) + \boldsymbol{c}(t-s), \qquad t^* = t + a, \tag{4}$$

we have, by the use of the objectivity condition (3), that

$$\begin{aligned} T^*(X,t^*) &= Q(t)\,T(X,t)\,Q(t)^T \\ &= Q(t)\,\mathcal{F}_\phi(\chi^t, X, t)\,Q(t)^T \\ &= Q(t^*-a)\,\mathcal{F}_\phi\big((Q^{t^*-a})^T((\chi^t)^* - \boldsymbol{c}^{t^*-a}) + \boldsymbol{x}_0,\ X,\ t^*-a\big)\,Q(t^*-a)^T \\ &:= \mathcal{F}_{\phi^*}\big((\chi^t)^*, X, t^*\big). \end{aligned} \tag{5}$$

Therefore, we have defined the corresponding response function for the observer ϕ^* as

$$T^*(X,t^*) = \mathcal{F}_{\phi^*}\big((\chi^t)^*, X, t^*\big), \qquad X \in \mathcal{B}, \tag{6}$$

given by (5) uniquely, for any past history of motion $(\chi^t)^*(Y,s) = \chi^*(Y, t^*-s)$. Note that in general, the response function $\mathcal{F}_{\phi^*}$ depends on the change of observer from ϕ to ϕ^* (characterized by $Q(t)$, $\boldsymbol{c}(t)$, $\boldsymbol{x}_0$, and a) in an explicit manner given in (5).

---

[3] For non-objective constitutive quantities (e.g., the total energy), since they are themselves observer-dependent, it would be absurd to require observer invariance for their response functions. Only their objective part (e.g., the internal energy) should be required to be independent of observers.

[4] I admit my misunderstanding concerning Footnote 1 in [2] on the interpretation of MFI in Chap. 6 of [8].

[5] There is no need to introduce a reference configuration in this formulation as demanded in (2.5) of [3]. The formulation in referential description is treated in Sect. 5 of [2].

Concerning (2) and (6), both Murdoch (e.g. (3.5) in [1]) and Liu ((3.4) in [2]) agree that the condition of objectivity can be expressed as

$$\mathcal{F}_{\phi^*}\big((\mathcal{X}^t)^*, X, t^*\big) = Q(t)\,\mathcal{F}_\phi(\mathcal{X}^t, X, t)\,Q(t)^T, \qquad X \in \mathcal{B}, \tag{7}$$

for any motion history $\mathcal{X}^t$ and its corresponding $(\mathcal{X}^t)^*$ related by (4). While Murdoch regards it as an invocation of the requirements O.4 and O.5, Liu regards this condition as a (mathematical) definition of the response function for any other observer once it is given for one observer.

According to Liu's interpretation, from the *mathematical* point of view, it is redundant to make the assumption O.4 for observer agreements on the nature of ideal materials (in Murdoch's interpretation of O.4) that the corresponding response function should depend on the same corresponding list of variables (see (2.26) and (2.27) in [1]). The argument that because the observers may not be in *communication* with each other, it is impossible to mandate the response functions for other observers from the response function chosen by a given observer, is somehow misleading. It is meaningless to talk about a change of observer, if there does not exist a known relation (in prior agreements of space-time and perhaps other dimensional considerations) between the two observers, irrespective of whether they are in *communication* or not in the *real world*.

4.1 The counter-example

In order to illustrate this point, we shall return to the response functions given in the counter-example of [2]. Consider the constitutive relation of a material body relative to the observer ϕ given by

$$T(X,t) = \hat{T}(D) = \lambda(\boldsymbol{e}\cdot D\boldsymbol{e})I + \mu D, \tag{8}$$

where λ and μ are material constants, I is the identity tensor, D is the symmetric part of the velocity gradient, and $\boldsymbol{e}$ is a constant unit vector. Now consider a different observer ϕ^*, related to the observer ϕ by the relative motion associated with the orthogonal transformation $Q(t)$. Then we have (from (4)) $D^* = QDQ^T$ and from the objectivity condition (3), it follows that

$$\begin{aligned} T^*(X,t^*) &= QT(X,t)Q^T = Q\hat{T}(D)Q^T \\ &= Q\big(\lambda(\boldsymbol{e}\cdot D\boldsymbol{e})I + \mu D\big)Q^T \\ &= \lambda\big(\boldsymbol{e}\cdot(Q^TD^*Q)\boldsymbol{e}\big)QQ^T + \mu\, Q(Q^TD^*Q)Q^T \\ &= \lambda(Q\boldsymbol{e}\cdot D^*Q\boldsymbol{e})I + \mu D^*. \end{aligned}$$

Therefore, we define from the above relation that

$$T^*(X,t^*) = \hat{T}^*(D^*) := \lambda(\boldsymbol{e}^*\cdot D^*\boldsymbol{e}^*)I + \mu D^*, \tag{9}$$

where $\boldsymbol{e}^* = Q(t)\boldsymbol{e}$ depends on the change of observer.

Note that the response function $\hat{T}$ is a function of D only because $\boldsymbol{e}$ is a fixed vector, and that $\boldsymbol{e}^*$ is not a fixed vector simply reflects the observer-dependence of the corresponding function $\hat{T}^*(D^*)$. It is obvious that the material objectivity condition (i.e., the standard restriction)

$$\hat{T}(QDQ^T) = Q\,\hat{T}(D)\,Q^T, \tag{10}$$

is not satisfied. Therefore, MFI is not satisfied and such a model does not make physical sense.

4.2 Murdoch's criticism on the counter-example

Nevertheless, the criticism in [3] employs a *different* interpretation of response functions, based on observer consensus (codified as O.1-5), by regarding the vector $\boldsymbol{e}$ and $\boldsymbol{e}^*$ also as constitutive variables for ϕ and ϕ^* (see (3.1) and (3.6) of [3]) respectively, so that

$$\begin{aligned} T(X,t) &= \tilde{T}(D,\,\boldsymbol{e}) &&= \lambda(\boldsymbol{e}\cdot D\,\boldsymbol{e})I + \mu D, \\ T^*(X,t^*) &= \tilde{T}^*(D^*,\,\boldsymbol{e}^*) &&= \lambda(\boldsymbol{e}^*\cdot D^*\,\boldsymbol{e}^*)I + \mu D^*. \end{aligned} \tag{11}$$

Then it is trivial to verify that

$$\begin{aligned} \tilde{T}^*(D^*, \boldsymbol{e}^*) &= Q\,\tilde{T}(D, \boldsymbol{e})\,Q^T, \\ \tilde{T}(QDQ^T, Q\boldsymbol{e}) &= Q\,\tilde{T}(D, \boldsymbol{e})\,Q^T, \end{aligned} \qquad \forall\, Q \in \mathcal{O}. \tag{12}$$

"Accordingly, the material in question is consistent with all objective considerations O.1-5. *Of course, this is a matter quite distinct from whether such a material exists*" (as stated in [3], Sect. 3.2) – We emphasize that the considerations O.1-5 do not require that the standard restriction be satisfied.

Indeed, the first equation of (12) is the condition of objectivity, and the second equation is regarded as the so-called *standard restriction* (see (3.2) in [3]) – Therefore, the model would no longer be a counter-example as interpreted by Liu. Nevertheless, the fact that the response function depends on the rotation of the observer frame seems to bother Murdoch in whether this could be regarded as a sensible physical model, even though all his requirements of objective considerations are satisfied.

In some cases, such as the kinetic gas [9] and turbulence [10,11] , it has been suggested that the *spin tensor* of the observer frame relative to the inertial frame be allowed in an objective combination of the constitutive variables. Indeed, it would be justifiable if a change of frame involved only Galilean transformations, even though we have not adopted this viewpoint in the present discussion. The situation is similar in this example. However, the allowance of the *rotation* of the observer frame in the response function, on the other hand, would hardly have any reasonable justifications.

Now, let us return to the example given in (11) to see whether the *standard restriction* is really satisfied. Let us represent the response functions in the alternative form suggested by Murdoch in (3.7) of [3]:

$$\begin{aligned} T(X,t) &= \bar{T}(D,\,\phi;\,\boldsymbol{e}) &&= \hat{T}(D,\,1;\,\boldsymbol{e}) &&= \lambda(\boldsymbol{e}\cdot D\,\boldsymbol{e})I + \mu D, \\ T^*(X,t^*) &= \bar{T}(D^*,\phi^*;\boldsymbol{e}) &&= \hat{T}(D^*,Q;\boldsymbol{e}) &&= \lambda(Q\boldsymbol{e}\cdot D^*\,Q\boldsymbol{e})I + \mu D^*. \end{aligned} \tag{13}$$

This is indeed a very clear and unambiguous notation indicating the explicit dependence of the observer frame. The objectivity condition can be easily verified, i.e.,

$$\hat{T}(D^*, Q; \boldsymbol{e}) = \hat{T}(QDQ^T, Q; \boldsymbol{e}) = Q\,\hat{T}(D, 1; \boldsymbol{e})Q^T.$$

This is the same as the conditions (12), in which like the first condition, the second one is nothing but the same condition of objectivity. It is *not* the so-called standard restriction as it appears to be.

On the other hand, that *the response function is the same for all observers* would require that

$$\bar{T}(\,\cdot\,, \phi;\,\cdot\,) = \bar{T}(\,\cdot\,, \phi^*;\,\cdot\,) \quad \text{or} \quad \hat{T}(\,\cdot\,, 1;\,\cdot\,) = \hat{T}(\,\cdot\,, Q;\,\cdot\,), \tag{14}$$

which combines with the condition of objectivity leads to the condition of material objectivity – *the* standard restriction. It can be written as either one of the following conditions (note that the first one is nothing but the condition of material objectivity (10) in Liu's formulation):

$$\begin{aligned} \hat{T}(QDQ^T, 1; \boldsymbol{e}) &= Q\,\hat{T}(D, 1; \boldsymbol{e})Q^T, \\ \hat{T}(QDQ^T, Q; \boldsymbol{e}) &= Q\,\hat{T}(D, Q; \boldsymbol{e})Q^T. \end{aligned} \tag{15}$$

One can easily check that *neither* of these conditions is satisfied by the response functions given in (13). Therefore, the material model given in (13) is indeed a counter-example since it is consistent with all the objectivity consideration O.1-5 (as stated before), but it does not satisfies the standard restriction.

4.3 Free choice of reference configuration for any observer?

In Sect. 3.4 of [3], the example of a transversely isotropic elastic material for the observer ϕ given by

$$T = \mathcal{H}(F) = s_0 I + s_1 F F^T + s_2 F\boldsymbol{n} \otimes F\boldsymbol{n} \tag{16}$$

is used to see how it is considered from the viewpoint of [1]. The purpose is to amplify the tenet of O.4 and O.5 for the existence of some other possibilities of the response function for the other observer ϕ^*, besides the one that can be determined from the condition of objectivity given in [2] by

$$T^* = \mathcal{H}^*(F^*) = s_0 I + s_1 F^* F^{*T} + s_2 F^*\boldsymbol{n}^* \otimes F^*\boldsymbol{n}^*. \tag{17}$$

Such possibilities do not exist for the following reasons:

The transformation H in $F^* = QFH^{-1}$ (see (3.12) of [3]), being the gradient of the mapping between the *corresponding* reference configurations for ϕ and ϕ^* at some instant must be an orthogonal transformation (as seen from Fig. 2 in [12] and Fig. 1 in [13]). Therefore, the polar decomposition of H should give $V = I$ in (3.21) of [3]. Accordingly, it implies that there are no possibilities other than the one given by (17).

The other non-trivial possibilities given in [3] are consequences of the *assumption* that any observer can have "his/her choice of reference configuration" (see [3], Sect. 2.3).[6] In other words, two observers can arbitrarily choose their reference configurations "(unknown to [them] if they are not in communication)". Accordingly, the transformation H is an invertible linear map in general and not necessarily an orthogonal transformation.

However, from this example, we can see that arbitrary choice of reference configuration by each observer is clearly inconsistent with the condition of objectivity. Indeed, the condition (3.20) of [3],

$$s_1\, I + s_2\, \boldsymbol{n}\otimes\boldsymbol{n} = H^{-1}(s_1^*\, I + s_2^*\, \boldsymbol{m}^*\otimes\boldsymbol{m}^*)H^{-T},$$

which is a consequence of objectivity for all possible responses (the assumption O.5), can *not* be satisfied by *any* invertible H for given s_1 and s_2.[7] Therefore, the choice of the two reference configurations can *not* be arbitrary. The very effort of finding the solutions of this equation for possible non-orthogonal transformation H is a denial of the assumption of free choice of reference configuration. Furthermore, the non-trivial solutions are *restrictions* (depending on material properties) imposed on the possible choices of reference configurations for a pair of observers with respect to each other. Consequently, besides the usual prior agreements on space-time and other physical units, without *actual* communication between the observers it would be impossible to make the right choice of reference configurations, because it may depend on the material properties of the other observer (see e.g., Case 2: for $V = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_2^2 = s_1\lambda_1^2/(s_1 + s_2)$ in Sect. 3.4 of [3]).

5 Final remarks

The essential idea quoted at the very beginning that "the response of a material is the same for all observers" has not been clearly formulated by Murdoch. It is entailed in the requirements of observer consensus as the assumptions O.4 and O.5. In particular, the assumption O.4 – agreement upon "the nature of any given ideal material", is vaguely interpreted as the requirement that response functions with respect to different observers must depend on *the same corresponding list* of variables (see (2.26) and (2.27) in [1]). However, occasionally, it may require some reinterpretations to the choice of the "same corresponding list", as we have seen in the example with response functions (8), (9) and their reinterpretations in (3.1) and (3.6) of [3] mentioned in Sect. 4.2. Furthermore, it may also require some physical intuition in order to judge whether a hypothetical proposed material model would be one that makes physical sense.

Take as another example, the explicit dependence of the response function on current position of the motion $\boldsymbol{x} = \chi^t(X, 0)$ and time t in (2). It can be proved that such dependence is not allowed from the condition of material objectivity – i.e., the condition (7) together with $\mathcal{F}_{\phi^*} = \mathcal{F}_\phi$ (see e.g., [5,7]). However, it cannot be proved by the objectivity considerations O.1-5 alone[8] (including the condition of objectivity (7) but not the standard restrictions), and perhaps, in order to exclude the possible dependence as one would expect, some other arguments would have to be conceived. Arguments such as these (either right or wrong is irrelevant here), not only require some physical intuitions purposely conceived for occasional cases, but also show that the assumption O.4 cannot be stated unambiguously in mathematical terms. Furthermore, the generality of a principle cannot be proved merely by examples, such as elastic materials, viscous fluids, and some others considered in [1].

Therefore, regarding the viewpoint of objectivity considerations O.1-5 as a guiding principle in continuum physics, besides our disagreement, especially on the assumption, that it is possible to contemplate *arbitrary* relative motions between two *given* observers, used as his essential arguments in the

[6] It is interesting to note that, in [12], this assumption is not adopted, as depicted in Fig. 2 and the expression (2.9).

[7] It is easier to see from the example: $T = s_1B$ and $T^* = s_1^*B^*$. The objectivity requires that $s_1^*B^* = s_1QBQ^T$, which cannot be satisfied for *any* invertible H. Note that $B^* = QFH^{-1}H^{-T}F^TQ^T \neq QBQ^T$ unless H is orthogonal.

[8] See Sect. 5.1 concerning Remark 9 of [3]

proofs, it is impractical, because relying on physical arguments and intuitions rather than mathematical reasoning run against the rational spirit deeply cherished in Modern Continuum Mechanics.

On the other hand, as we have pointed out that corresponding to any response function proposed for one observer, the response function for any other observer can be defined through the objectivity condition uniquely, therefore, in practice, it is the condition of material objectivity (referred to as standard restrictions in [1]) that imposes restrictions on the candidate response functions, and more importantly, with simple mathematical reasoning, it is capable of delineating classes of physically legitimate materials, in the sense that the response of a material is the same for all observers.

5.1 Remark on explicit dependence of position and time

In response to the criticism concerning the explicit dependence of the current position and time mentioned above, an analysis (as Remark 9) is added in [3]. It considers response functions of the form (3.28)[9] and (3.29) relative to observer O and O^* respectively (the numbering referred to the equations in [3]),

$$T(\boldsymbol{x},t)=\hat{T}(\boldsymbol{x},t),\qquad T^*(\boldsymbol{x}^*,t^*)=\breve{T}^*(\boldsymbol{x}^*,t^*).$$

The objectivity condition (2.2) requires that the relation (3.33) holds,

$$T^*(\boldsymbol{x}^*,t^*)=Q(t)T(\boldsymbol{x},t)Q(t)^T.$$

The purpose is to prove that the response functions $\hat{T}$ and $\breve{T}^*$ must be independent of current position $\boldsymbol{x}$ and time t.

The analysis in [3] employs the *assumption* of arbitrary relative motions between observers O and O^* and from the objectivity condition (3.31)[10] proves that both $\hat{T}$ and $\breve{T}^*$ must be constant functions, which is a contradiction to (3.33) since T^* depends on $Q(t)$ yet is constant in time. If this argument is correct[11], then the *assumption* must be false and independence of position and time has not been proved. It is interesting to note that this conclusion seems to disclaim the validity of arbitrary relative motions between observer pairs.

We understand that the above conclusion could not possibly be appreciated by Murdoch who regards *the* assumption as a valid one in the first place. Therefore, in [3], it concludes from this contradiction that "material response of the form (3.28) is ruled out".

Independent of whatever conclusions, the main disagreement between Murdoch and Liu comes down to the same arguments as before – the validity of the assumption that it is possible to contemplate arbitrary relative motions between two observers, a key issue of the present dispute.

5.2 Remark on "a key modelling issue"

In Sect. 4 of [3], it states that "constitutive relations $\cdots$ can be shown (by the reasoning of [1], which invokes O.4 and O.5 and accordingly *different hypothetical relative motions of observer pairs*) to imply *isrbm* $\cdots$". Indeed, Murdoch employs the key *assumption* that it is possible to have different relative motions between two given observers in his proofs of the above statement and his claim that "Liu's result [the condition of material objectivity] follows from O.4 and O.5 without [the form-invariance] assumption (2.4)" (from (a) of Sect. 2.3 in [3]), as well as many other related conclusions, such as the one concerning the explicit dependence of response function on the current position and time, and (iii), (iv) of Concluding Remarks in Sect. 6 of [3].

[9] We do not suggest response function of the form (3.28), instead, we mean the explicit dependence of $\boldsymbol{x}$ and t among other quantities associated with the motion history as given in (2) of Sect. 4. Nevertheless, the simple example (3.28) serves the purpose of this discussion.

[10] Note that (3.31) establishes the relation between the two functions $\hat{T}$ and $\breve{T}^*$ for *the* two (given) observers O and O^* respectively, similar to the relation (5) in Sect. 4.

[11] In fact, it is not necessary a contradiction to (3.33), since it implies that $T=T^*=\alpha I$ for some constant α, which would have proved the independence of x and t.

Let us consider a simple situation. Suppose that one of the observer is in a fixed position and the other is on a moving train, it is obvious that the relative motion between them is nothing but the motion of the train. Therefore, we strongly disagree with this assumption and consequently, the above statement that O.4 and O.5 imply *isrbm* and the other related conclusions mentioned above are not valid from our viewpoint, as we have discussed hitherto, with counter-examples and other arguments.

Furthermore, citing [3] those theories in the literature, for which the constitutive relations involve observer's spin, as evidence for the violation of *isrbm* is misleading. Indeed, as we have commented in Sect. 4.2, the spin dependence does not necessarily violate *isrbm*, if Galilean transformations, instead of Euclidean transformations, are invoked in a change of observer. The question of whether Euclidean or Galilean transformations should be invoked in a change of observer must be verified by rigorous experimental results. Therefore, it is misleading to claim that *isrbm* is questionable because it "imposes an *a prioi* restriction upon Nature" due to lacks of theoretical support from some spin-dependent theories.

Acknowledgement: *The author gratefully acknowledges the support of CNPq-Brazil, through the Research Fellowship - PQ.*

References

1. Murdoch, A. I.: Objectivity in classical continuum physics: a rationale for discarding the 'principle of invariance under superposed rigid body motions' in favour of purely objective considerations, *Continuum Mech. Thermodyn.* **15**, (2003) 309-320.
2. Liu, I-Shih: On Euclidean objectivity and the principle of material frame-indifference, *Continuum Mech. Thermodyn.* **16**, (2004) 177-183.
3. Murdoch, A. I.: On criticism of the nature of objectivity in classical continuum physics, *Continuum Mech. Thermodyn.* (2004).
4. Truesdell, C.; Noll, W.: *The Non-Linear Field Theories of Mechanics*, Handbuch der Physik, edited by S. Flügge, Vol. III/3, Springer-Verlag, Berlin-Heidelberg (1965).
5. Truesdell, C.: *A First Course in Rational Continuum Mechanics*, Vol. 1, Second Edition, Academic Press, Boston (1991).
6. Noll, W.: Lectures on the foundations of continuum mechanics and thermodynamics, *Arch. Rational Mech. Anal.* **52**, (1973) 62-92.
7. Liu, I-Shih; *Continuum Mechanics*, Springer, Berlin-Heidelberg-New York (2002).
8. Šilhavý, M.: *The Mechanics and Thermodynamics of Continuous Media*, Springer, Berlin-Heidelberg (1997).
9. Müller, I.: On the frame dependence of stress and heat flux. *Arch. Rational Mech. Anal.* **45**, (1972) 241-250.
10. Speziale, C. G.: A review of material frame-indifference in mechanics, *Appl. Mech. Rev.* **51** (1998) 489-504.
11. Sadiki, A.; Bauer, W.; Hutter, K.: Thermodynamically consistent coefficient calibration in nonlinear and anisotropic closure models for turbulence, *Continuum Mech. Thermodyn.* **12** (2000) 131-149.
12. Murdoch, A. I.: On objectivity and material symmetry for simple elastic solids, *J. Elasticity* **60**, (2000) 233-242.
13. Liu, I-Shih: On the transformation property of the deformation gradient under a change of frame, *J. Elasticity* **71**, (2003) 73-80.