# On Euclidean objectivity and the principle of material frame-indifference

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**Abstract.** The essential idea of the principle of material frame-indifference — material properties are frame-indifferent, or constitutive functions are form-invariant relative to change of observers — has been disputed recently by Murdoch in [1,2] by the claim that the standard restrictions on constitutive functions can be deduced by purely objective considerations, without the requirement of form-invariance of constitutive functions. The purpose of this paper is to show that such a claim is groundless by pointing out blunders in the specious proofs and by presenting counter-examples.

**Key words:** constitutive functions, change of frame, material objectivity
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## 1 Introduction

The principle of material frame-indifference (MFI) plays an important role in the development of continuum mechanics (see Truesdell and Noll 1965 [3]) by delivering restrictions on the formulation of constitutive functions of material bodies. It is embedded in the idea that material properties should be independent of observations made by different observers. Since different observers are related by a time-dependent rigid transformation, known as Euclidean transformation, material frame-indifference is sometimes interpreted as invariance under superposed rigid body motions.

On the other hand, Euclidean objectivity of an observable quantity is an invariance property concerning its transformation behavior under a change of frame. The concept of Euclidean objectivity has nothing to do with material property. Therefore, Euclidean objectivity and the principle of material frame-indifference are two independent concepts. Essentially, the principle of MFI merely states that the constitutive function of an 'objective' quantity should be independent of observers. Although this point of view may have been obscured sometimes, the form-invariance of constitutive functions has always been assumed by almost everyone (see [3]-[11]) — however, this has been challenged recently by Murdoch in [1,2].

In [1], Murdoch claimed to have proved that purely objective considerations are sufficient to mandate *standard restrictions* upon constitutive functions, and restrictions are shown to involve only *proper* orthogonal transformations. Consequently, "the 'principle' [of invariance under superposed rigid body motions] serves no useful purpose and should be discarded" – this would be a truly remarkable result if it were true. A similar claim was reported earlier in [2].

Despite occasional confusions and considerable controversy over the years, that a principle of fundamental importance has been pronounced superfluous deserves a more careful scrutiny. By examining

the arguments, one can find that the main passages of the proofs in [1,2] rely on an erroneous interpretation concerning observers and relative motions. Indeed, there is a well-known counter-example, the case of a kinetic gas, first pointed out by Müller [12], and other counter-examples are easy to construct.

Moreover, the question of whether the Euclidean transformation should involve general or only proper orthogonal tensors has always been regarded as a question of definition of choice and should only be decided by experiments. This has been disputed also in [1,2] from its claim that only proper orthogonal transformations are *proved* to be allowed. The erroneous proofs also invalidate such a claim.

## 2  Frame of reference and Euclidean objectivity

The event world or space-time $\mathcal{W}$ of continuum mechanics (see [4]) can be mapped onto the product space of a three-dimensional Euclidean space $\mathcal{E}$ and the set of real numbers $I\!\!R$ through a one-to-one mapping,

$$\phi : \mathcal{W} \to \mathcal{E} \times I\!\!R.$$

Such a mapping is called a frame of reference. Let us denote by $\mathcal{W}_t$ the totality of simultaneous events at the instant $t$, and $\phi_t$ the restriction of $\phi$ to $\mathcal{W}_t$, so $\phi_t : \mathcal{W}_t \to \mathcal{E}$ associates the placement of an event with a location in the Euclidean space $\mathcal{E}$.

A body $\mathcal{B}$ is a set of material points, and we shall place it, through a one-to-one mapping, with a region in $\mathcal{E}$ relative to a frame of reference. Such an identification is called a configuration of the body. A motion of the body is a continuous sequence of configurations in time. More specifically, we consider

$$\mathcal{B} \xrightarrow{\bar{\chi}_t} \mathcal{W}_t \xrightarrow{\phi_t} \mathcal{E}, \qquad \boldsymbol{x} = \phi_t\big(\tilde{\chi}_t(X)\big) = \tilde{\chi}_{\phi_t}(X);$$

then $\tilde{\chi}_{\phi_t} = \phi_t \circ \tilde{\chi}_t$ is a motion of the body $\mathcal{B}$ relative to the frame of reference $\phi$. We shall write $\chi(\cdot, t) = \tilde{\chi}_{\phi_t}(\cdot)$, and hence denote the motion as $\chi(X, t)$,

$$\chi : \mathcal{B} \times I\!\!R \to \mathcal{E}, \qquad \boldsymbol{x} = \chi(X, t).$$

Let $\phi$ and $\phi^*$ be two frames of reference. We call

$$\phi^* \circ \phi^{-1} : \mathcal{E} \times I\!\!R \to \mathcal{E} \times I\!\!R$$

a change of frame from $\phi$ to $\phi^*$. In general, the change of frame which maps $(\boldsymbol{x}, t)$ to $(\boldsymbol{x}^*, t^*)$ is a Euclidean transformation given by

$$\boldsymbol{x}^* = Q(t)(\boldsymbol{x} - \boldsymbol{x}_o) + \boldsymbol{c}(t), \qquad t^* = t + a, \tag{1}$$

for some $a \in I\!\!R$, $\boldsymbol{x}_o \in \mathcal{E}$, $\boldsymbol{c}(t) \in \mathcal{E}$, and $Q(t) \in \mathcal{O}$, where $\mathcal{O}$ is the group of orthogonal transformations on the translation space $V$ of the Euclidean space $\mathcal{E}$. In particular, $* := \phi_t^* \circ \phi_t^{-1} : \mathcal{E} \to \mathcal{E}$ is given by

$$\boldsymbol{x}^* = \phi_t^*(\phi_t^{-1}(\boldsymbol{x})) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_o) + \boldsymbol{c}(t). \tag{2}$$

The change of frame $*$ on the Euclidean space $\mathcal{E}$ gives rise to a linear mapping on the translation space $V$ of $\mathcal{E}$ in the following way: Let $\boldsymbol{v} = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in V$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{E}$ in the frame $\phi$, and $\boldsymbol{v}^* = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in V$ be the corresponding difference vector in the frame $\phi^*$. Then from (2), it follows immediately that

$$\boldsymbol{v}^* = Q(t)\boldsymbol{v}.$$

Any vector quantity in $V$, which has this transformation property, is said to be objective with respect to Euclidean transformations. Euclidean objectivity can be extended to any tensor spaces of $V$. In particular, we say that a scalar $s$, a vector $\boldsymbol{v}$, or a tensor $T$ is a Euclidean objective (or simply objective) quantity, if

$$s^* = s, \qquad \boldsymbol{v}^* = Q(t)\boldsymbol{v}, \qquad T^* = Q(t)TQ(t)^T.$$

Let $\chi(X, t)$ be a motion of the body in the frame $\phi$, and $\chi^*(X, t^*)$ be the corresponding motion in $\phi^*$. Then from (2), we have

$$\chi^*(X, t^*) = Q(t)\big(\chi(X, t) - \boldsymbol{x}_o\big) + \boldsymbol{c}(t), \qquad X \in \mathcal{B}. \tag{3}$$

Consequently, the transformation property of a kinematic quantity, associated with positions of the motion in $\mathcal{E}$, can be derived from (3). In particular, one can easily show that the velocity and the

velocity gradient are not objective quantities; while the surface normal and the symmetric part of the velocity gradient are objective quantities (for general discussions of kinematic quantities and their transformation properties, please see [3]–[7]).

On the other hand, the transformation properties of physical quantities, such as mass density, energy, force, temperature etc., cannot be derived in a change of frame, and their objectivity are usually postulated. For example, mass density, internal energy (but not the total energy), body force, surface traction, and temperature are regarded as objective physical quantities.

## 3  Constitutive relations

The most important aspect of changes of frame lies in the formulation of constitutive functions. For simplicity, we shall consider only mechanical theory.

Consider a motion $\chi$ relative to the frame $\phi$ and let $T(X, t)$ be the value of the Cauchy stress tensor at the material point $X$ and time $t$ in the frame $\phi$. We postulate the constitutive relation in the following form,

$$T(X,t) = \mathcal{F}_\phi(\chi^t, X), \qquad X \in \mathcal{B}, \tag{4}$$

where $\mathcal{F}_\phi$ is a functional and the argument function $\chi^t : \mathcal{B} \times [0, \infty) \to \mathcal{E}$ stands for the past history of the motion up to the instant $t$, $\chi^t(Y, s) = \chi(Y, t - s)$ for $Y \in \mathcal{B}$ and $0 \le s < \infty$. Similarly, relative to the frame $\phi^*$, the corresponding constitutive relation can be written as

$$T^*(X, t^*) = \mathcal{F}_{\phi^*}\big((\chi^t)^*, X\big), \qquad X \in \mathcal{B}, \tag{5}$$

where $(\chi^t)^*(Y, s) = \chi^*(Y, t^* - s)$ and from (3), we have

$$(\chi^t)^*(Y, s) = Q(t - s)\big(\chi^t(Y, s) - \boldsymbol{x}_o\big) + \boldsymbol{c}(t - s). \tag{6}$$

### 3.1  Euclidean objectivity

Since surface traction and surface normal are objective vectors, it follows that the Cauchy stress is an objective tensor, i.e., $T^*(X, t^*) = Q(t)T(X, t)Q(t)^T$. Therefore, from (4) and (5), it follows that

$$\mathcal{F}_{\phi^*}\big((\chi^t)^*, X\big) = Q(t)\mathcal{F}_\phi(\chi^t, X)Q(t)^T, \qquad X \in \mathcal{B}, \tag{7}$$

for any motion history $\chi^t$ and where $Q(t)$ is the orthogonal part of the change of frame from $\phi$ to $\phi^*$. This will be referred to as the condition of Euclidean objectivity, or simply, the objectivity condition.[1] Note that this condition has nothing to do with the material nature of the function $\mathcal{F}_\phi$ itself. In fact, it can be regarded as the definition of $\mathcal{F}_{\phi^*}$, once the constitutive function $\mathcal{F}_\phi$ is given.

### 3.2  Principle of material frame-indifference

Since any intrinsic property of materials should be independent of frame of reference, it is required that for any objective quantity, the constitutive function must be invariant with respect to any change of frame. Mathematically, the principle can be stated as follows:

*The constitutive function of an objective quantity must be independent of the frame, i.e.,* $\mathcal{F}_\phi(\cdot, X) = \mathcal{F}_{\phi^*}(\cdot, X)$ *for any frames of reference* $\phi$ *and* $\phi^*$.

This is also referred to as form-invariance of constitutive functions (see [11]).

Combining the form-invariance with the condition of Euclidean objectivity (7), we obtain

$$\mathcal{F}\big((\chi^t)^*, X\big) = Q(t)\mathcal{F}(\chi^t, X)Q(t)^T, \qquad \forall\, Q(t) \in \mathcal{O}, \ X \in \mathcal{B}, \tag{8}$$

and for any motion history $\chi^t$. Since this condition involves only the function $\mathcal{F} = \mathcal{F}_\phi = \mathcal{F}_{\phi^*}$, it imposes restrictions upon the constitutive function $\mathcal{F}$ for a legitimate material model (referred to as the 'standard restrictions' in [1]). It is usually referred to as the condition of *material* objectivity, to impart its relevance in characterizing material property.

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[1]  In [1], Murdoch regarded this as the 'one motion, two observers' aspect of objectivity, and he called it 'invariance under superposed *virtual* rigid body motions'. This is sometimes misinterpreted as a version of the principle of material frame-indifference (see e.g., Chap. 6 in [8]).

Moreover since $(\chi^t)^*$ and $\chi^t$ can be regarded as relative rigid body motions related by (6), the material objectivity condition (8) has been postulated as a different version of MFI, known as the *principle of invariance under superposed rigid body motions* from the perspective of a single observer (abbreviated in [1] to 'isrbm', in the sense of 'one observer, two motions').

One can easily verify that the three concepts – (i) Euclidean objectivity of a constitutive quantity, (ii) the form-invariance of constitutive functions (equivalently, the principle of MFI), and (iii) the principle of invariance under superposed rigid body motions – are not mutually independent, and any two of them imply the third as pointed out in [11].

## 4 Euclidean objectivity vs. material objectivity

It is claimed in [1,2] that the (Euclidean) objectivity condition (7) implies the 'standard restrictions', i.e., the condition of material objectivity (8), without invoking the form-invariance of the principle of MFI. Therefore, only Euclidean objectivity is relevant in the formulation of constitutive theories of material bodies, and Murdoch regarded the principle of 'isrbm' (the version of MFI in the sense of 'one observer, two motions'), as a red herring: it is "neither physical nor necessary". But is it true? Let us give a counter-example and then look at the proofs in [1,2] to see where they went wrong.

### 4.1 A counter-example

Consider the constitutive relation (4) of a material body in the frame $\phi$ given by

$$T(X,t) = \hat{T}(D) = \lambda(\boldsymbol{e} \cdot D\boldsymbol{e})I + \mu D, \tag{9}$$

where $\lambda$ and $\mu$ are material constants, $I$ is the identity tensor, $D$ is the symmetric part of the velocity gradient, and $\boldsymbol{e}$ is a constant unit vector. Both $D$ and $\boldsymbol{e}$ are objective quantities. The corresponding constitutive relation (5) in the frame $\phi^*$ is given by

$$T^*(X,t^*) = \hat{T}^*(D^*) = \lambda(\boldsymbol{e}^* \cdot D^*\boldsymbol{e}^*)I + \mu D^*. \tag{10}$$

Note that the constitutive function $\hat{T}^*$ depends "explicitly" on the frame, since $\boldsymbol{e}^* = Q(t)\boldsymbol{e}$, where $Q(t) \in \mathcal{O}$ is associated with the change of frame from $\phi$ to $\phi^*$.

Since, $D^* = QDQ^T$ and $\boldsymbol{e}^* = Q\boldsymbol{e}$,

$$\hat{T}^*(D^*) = \lambda\,(Q\boldsymbol{e}) \cdot (QDQ^T)(Q\boldsymbol{e})I + \mu QDQ^T = Q\big(\lambda(\boldsymbol{e} \cdot D\boldsymbol{e})I + \mu D\big)Q^T = Q\hat{T}(D)Q^T,$$

and hence it satisfies the objectivity condition (7). On the other hand,

$$\hat{T}(QDQ^T) = \lambda(\boldsymbol{e} \cdot (QDQ^T)\boldsymbol{e})I + \mu QDQ^T \neq Q\big(\lambda(\boldsymbol{e} \cdot D\boldsymbol{e})I + \mu D\big)Q^T = Q\hat{T}(D)Q^T.$$

Consequently, it does not satisfy the restriction of material objectivity (8) in the present case.

Therefore, although the relations (9) and (10), in $\phi$ and $\phi^*$ respectively, satisfy the objectivity condition, they cannot be adopted as constitutive relations of a legitimate material model. Note that they are not form-invariant. To satisfy the form-invariance, it would have to define, instead of (10),

$$T^*(X,t^*) = \hat{T}(D^*) = \lambda(\boldsymbol{e} \cdot D^*\boldsymbol{e})I + \mu D^*,$$

which in turn, does not satisfy the objectivity condition. A similar example for an elastic material is given in Sec. 5.

Another well-known counter-example, the case of a kinetic gas, first investigated by Müller [12], concerning constitutive equations with explicit presence of the spin of the frame, is shown to be Euclidean objective but does not satisfy the principle of MFI (see [11,12]).[2]

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[2] Apparently Murdoch did not regard this as a counter-example, because he considered the spin of the frame as a constitutive variable, so that objective combinations involving spin are allowed in the constitutive equations (see [13]).

### 4.2   The "proof" in [1]

Now let us examine the seemingly elegant "proof" in [1]. For simplicity, take $T(X,t) = \hat{T}\big(D(X,t)\big)$ and $T^*(X,t^*) = \hat{T}^*\big(D^*(X,t^*)\big)$ as an example. The proof goes like this (all examples considered in [1] follow the same pattern of reasoning):

From the objectivity condition (7),
$$\hat{T}^*\big(D^*(X,t^*)\big) = Q(t)\hat{T}\big(D(X,t)\big)Q(t)^T, \quad \text{where} \quad D^*(X,t^*) = Q(t)D(X,t)Q(t)^T. \tag{11}$$
Now consider "*two* possible relative motions" of $\phi$ and $\phi^*$ which correspond to the same state $(D^*)$ in $\phi^*$ "at the given instant" $t$. From the $\phi$ perspective there will be two different velocity gradients $D_1$ and $D_2$, related to $D^*$ via two different orthogonal-valued functions $Q_1(t)$ and $Q_2(t)$. Specifically,
$$Q_2 D_2 Q_2^T = D^* = Q_1 D_1 Q_1^T, \tag{12}$$
and from (11),
$$Q_2 \hat{T}(D_2) Q_2^T = \hat{T}^*(D^*) = Q_1 \hat{T}(D_1) Q_1^T. \tag{13}$$
Now let an orthogonal transformation be defined as
$$Q(t) = Q_2(t)^T Q_1(t).$$
Then from (12), $D_2 = Q D_1 Q^T$, and by writing $D_1$ as $D$, (13) implies
$$\hat{T}(QDQ^T) = Q\hat{T}(D)Q^T, \tag{14}$$
which establishes the condition of material objectivity (8) in the present case, without invoking the form-invariance of constitutive functions. $\square$

The proof is simple enough. However, let us examine the quoted expression, "two possible relative motions" of $\phi$ and $\phi^*$. Since by definition, any two observers (i.e., two frames of reference) are related by a (one and only one) relative motion of time-dependent rigid transformation, any two relative motions with respect to $\phi^*$ are necessarily associated with two different observers, say $\phi_1$ and $\phi_2$. In other words, by denoting the constitutive functions as $\hat{T}_1$ and $\hat{T}_2$ respectively in $\phi_1$ and $\phi_2$, relation (13) should read
$$Q_2 \hat{T}_2(D_2) Q_2^T = \hat{T}^*(D^*) = Q_1 \hat{T}_1(D_1) Q_1^T,$$
while from the transformation property of the velocity gradient, relation (12) is valid at any instant (therefore, emphasizing "at the given instant" in the proof is pointless). Consequently, conclusion (14) becomes
$$\hat{T}_2(QDQ^T) = Q\hat{T}_1(D)Q^T,$$
which is simply the objectivity condition (11) for the change of frame from $\phi_1$ to $\phi_2$, and the proof is void.

A similar version of the proof was also mentioned in [1], by considering a single relative motion of $\phi$ and $\phi^*$ and a "process" in which the values of $D^*$ are the same at distinct instants $t_1$ and $t_2$, i.e.,
$$Q(t_2)D(t_2)Q(t_2)^T = D^*(t_2^*) = D^*(t_1^*) = Q(t_1)D(t_1)Q(t_1)^T. \tag{15}$$
This is equivalent to (12), and consequently, the same conclusion (14) can also be obtained. And as a result of this conclusion, $Q = Q(t_2)^T Q(t_1)$ must be a proper orthogonal transformation. Indeed, since $\det Q(t) = \pm 1$ and by continuity $\det Q(t_1)$ and $\det Q(t_2)$ must have the same sign (in [1] a different reasoning was employed). Therefore only proper orthogonal transformations are allowed in the condition (14).

Of course, the restriction (14) should hold for any $D(t)$ as required by the principle of 'isrbm'. However, the above reasoning can *not* be true for any $D(t)$ in general, since the "process" which satisfies relation (15) at some instants $t_1$ and $t_2$ is only possible if $D(t_1)$ and $D(t_2)$ have the same eigenvalues. In other words, neither the principle of 'isrbm' and nor the claim that only proper orthogonal transformations are allowed has been proved by purely objective considerations.

### 4.3   The "proof" in [2]

A different "proof" was presented as a remark in [2]. Take the same example for simplicity (the original proof was for elastic materials[3]). The proof goes in a few steps:

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[3]  In retrospect, I found that the same proof was presented in [14].

Suppose that at the instants $t_1$ and $t_2$, the perceived views by the observers $\phi$ and $\phi^*$, are related by the corresponding $Q_1 = Q(t_1)$ and $Q_2 = Q(t_2)$, and let $Q = Q_2^T Q_1$. Then from the objectivity condition,

$$
\begin{aligned}
Q_1 \hat{T}(D_1) Q_1^T = \hat{T}^*(Q_1 D_1 Q_1^T) &= \hat{T}^*\big(Q_2(Q_2^T Q_1) D_1(Q_1^T Q_2) Q_2^T\big) \\
&= \hat{T}^*\big(Q_2(Q D_1 Q^T) Q_2^T\big) \ " = " \ Q_2 \hat{T}(Q D_1 Q^T) Q_2^T,
\end{aligned}
\tag{16}
$$

which implies the standard restriction of material objectivity,

$$
\hat{T}(Q D_1 Q^T) = Q \hat{T}(D_1) Q^T.
$$

The above restriction should hold for any $Q$ is a consequence of the arbitrariness of observer $\phi^*$. Moreover, by the argument as before, $Q$ must be a proper orthogonal transformation. $\square$

The proof is surprisingly simple. Now let us take a more careful look at the crucial passage indicated by " $=$ " in (16). Since the constitutive functions are not assumed to be form-invariant, we may write

$$
T(X, t) = \hat{T}\big(D(X, t), \phi_t\big), \qquad T^*(X, t^*) = \hat{T}^*\big(D^*(X, t^*), \phi_t^*\big),
$$

in which the frame-dependence is indicated explicitly, and the objectivity condition reads

$$
\hat{T}^*\big(Q(t) D(X, t) Q(t)^T, \phi_t^*\big) = Q(t) \hat{T}\big(D(X, t), \phi_t\big) Q(t)^T,
$$

which should hold for any $D(X, t)$ and for any change of frame from $\phi$ to $\phi^*$ associated with $Q(t)$. Moreover, it is important to note that the equation is evaluated at any instant $t$; in particular, both $Q(t)$ and $\phi_t$ must be evaluated at the *same* instant. Therefore, the following expression,

$$
\hat{T}^*\big(Q(t_2) D(X, t_1) Q(t_2)^T, \phi_{t_1}^*\big) = Q(t_2) \hat{T}\big(D(X, t_1), \phi_{t_1}\big) Q(t_2)^T,
$$

with $Q(t_2)$ and $\phi_{t_1}$ at different instants, is *not* valid in general (check this out from the counter-example). This is exactly what happens in the crucial passage in (16) and the proof is invalid.

## 5   Remarks on constitutive relations in referential description

In Sec. 3, we have formulated constitutive functions in the material description, i.e., they are regarded as functions of $(X, t)$ for $X \in \mathcal{B}$. Let us turn to the referential description by introducing a reference configuration of the body (see [9]).

Let $\tilde{\kappa} : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant $t_0$. Then $\kappa = \phi_{t_0} \circ \tilde{\kappa} : \mathcal{B} \to \mathcal{E}$ and $\kappa^* = \phi_{t_0}^* \circ \tilde{\kappa} : \mathcal{B} \to \mathcal{E}$ are the two corresponding reference configurations of $\mathcal{B}$ in the frames $\phi$ and $\phi^*$ at the same instant, and

$$
\boldsymbol{X} = \kappa(X) \in \mathcal{E}, \qquad \boldsymbol{X}^* = \kappa^*(X) \in \mathcal{E}, \qquad X \in \mathcal{B}.
$$

Let us denote by $\gamma = \kappa^* \circ \kappa^{-1}$ the change of reference configuration from $\kappa$ to $\kappa^*$ in the change of frame. Then it follows that $\gamma = \phi_{t_0}^* \circ \phi_{t_0}^{-1}$ and by (2), we have

$$
\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}(t_0),
\tag{17}
$$

where $K = \nabla_{\boldsymbol{X}} \gamma = Q(t_0)$ is a constant orthogonal tensor.

The motion in referential description relative to the change of frame is given by

$$
\boldsymbol{x} = \chi(X, t) = \chi\big(\kappa^{-1}(\boldsymbol{X}), t\big) = \chi_\kappa(\boldsymbol{X}, t), \qquad\qquad \chi = \chi_\kappa \circ \kappa,
$$

$$
\boldsymbol{x}^* = \chi^*(X, t^*) = \chi^*\big(\kappa^{*-1}(\boldsymbol{X}^*), t^*\big) = \chi_{\kappa^*}^*(\boldsymbol{X}^*, t^*), \qquad \chi^* = \chi_{\kappa^*}^* \circ \kappa^*.
$$

The constitutive functions in (4) and (5) can now be written as

$$
\mathcal{F}_\phi(\chi^t, X) = \mathcal{F}_\phi(\chi_\kappa^t \circ \kappa, X) := \mathcal{H}_\phi(\chi_\kappa^t, \boldsymbol{X}),
$$

$$
\mathcal{F}_{\phi^*}\big((\chi^t)^*, X\big) = \mathcal{F}_{\phi^*}\big((\chi_\kappa^t)^* \circ \kappa^*, X\big) := \mathcal{H}_{\phi^*}\big((\chi_\kappa^t)^*, \boldsymbol{X}^*\big).
$$

Since form-invariance requires $\mathcal{F}_\phi = \mathcal{F}_{\phi^*}$, it implies that the constitutive functions in the referential description are related by

$$
\mathcal{H}_{\phi^*}\big((\chi_\kappa^t)^*, \boldsymbol{X}^*\big) = \mathcal{H}_\phi\big((\chi_\kappa^t)^* \circ \gamma, \boldsymbol{X}\big),
$$

where $\gamma = \kappa^* \circ \kappa^{-1} = \phi_{t_0}^* \circ \phi_{t_0}^{-1}$. Therefore, they are not form-invariant in general, i.e., $\mathcal{H}_{\phi^*} \neq \mathcal{H}_\phi$, but rather they are related in the above manner.[4]

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[4] If the reference configuration is assumed to be unaffected by the change of frame (see p.308 of [3]) , then $\gamma$ is an identity map on $\mathcal{E}$, and $\mathcal{H}$ is form-invariant, $\mathcal{H}_{\phi^*} = \mathcal{H}_\phi$. Moreover, under this assumption, $K = I$.

Let $F = \nabla_{\boldsymbol{X}} \chi_\kappa$ and $F^* = \nabla_{\boldsymbol{X}^*} \chi_{\kappa^*}^*$ be the deformation gradients in $\phi$ and $\phi^*$, respectively. Then from (3) and (17), we obtain by the chain rule, $F^* = QFK^T$, which has also been derived in [1,2].

For elastic materials, by writing $T = \mathcal{H}(F)$ and $T^* = \mathcal{H}^*(F^*)$ in the change of frame from $\phi$ to $\phi^*$, one can easily verify the following conditions of Euclidean objectivity, material frame-indifference, and invariance under superposed rigid body motions (material objectivity condition), respectively:

$$\mathcal{H}^*(QFK^T) = Q\mathcal{H}(F)Q^T, \qquad \mathcal{H}^*(F^*) = \mathcal{H}(F^*K), \qquad \mathcal{H}(QF) = Q\mathcal{H}(F)Q^T. \tag{18}$$

As examples, let us consider two models of elastic bodies:

*Example 1.*
$$\mathcal{H}(F) = s_0 I + s_1 B + s_2 F\boldsymbol{n} \otimes F\boldsymbol{n},$$
$$\mathcal{H}^*(F^*) = s_0 I + s_1 B^* + s_2 F^*\boldsymbol{n}^* \otimes F^*\boldsymbol{n}^*.$$

*Example 2.*
$$\mathcal{H}(F) = s_0 I + s_1 B + s_2 B\boldsymbol{e} \otimes B\boldsymbol{e},$$
$$\mathcal{H}^*(F^*) = s_0 I + s_1 B^* + s_2 B^*\boldsymbol{e}^* \otimes B^*\boldsymbol{e}^*.$$

Here, $s_0, s_1, s_2$ are objective scalar material parameters, $B = FF^T$ is the left Cauchy-Green deformation tensor, while $\boldsymbol{n}$ in Example 1, is a fixed vector in the reference configuration $\kappa$, and $\boldsymbol{e}$ in Example 2, is a fixed vector in the frame $\phi$. They have the transformation properties: $B^* = QBQ^T$, $\boldsymbol{e}^* = Q\boldsymbol{e}$, and $\boldsymbol{n}^* = K\boldsymbol{n}$. Note that $K$ is the value of $Q$ at the instant $t_0$ associated with the reference configuration.

Example 1 is a legitimate material model, since it satisfies all the conditions in (18). Indeed, this material body is transversely isotropic[5] relative to the axis containing the vector $\boldsymbol{n}$ (see the example on p. 124 of [6]).

However, it is easy to verify that Example 2 satisfies the Euclidean objectivity condition $(18)_1$, but neither MFI condition $(18)_2$, nor the principle of superposed rigid body motions $(18)_3$. Consequently, it is not a legitimate model for any material body. This is another counter-example to the claims in [1,2].

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[5] A change of reference configuration $G$ is a symmetric transformation if it satisfies $\mathcal{H}(FG) = \mathcal{H}(F)$ (see e.g., [4]–[8]). In Example 1, any $G \in \mathcal{O}$, such that $G\boldsymbol{n} = \pm\boldsymbol{n}$ is a symmetric transformation.