

# A solid-fluid mixture theory of porous media

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## Abstract

The theories of mixtures in the framework of continuum mechanics have been developed throughout the sixties and seventies. In this paper, we shall focus on one of the typical constitutive theories obtained by Bowen [1] for mixture of elastic materials. We employ a different procedure in the exploitation of the entropy principle, by the use of Lagrange multipliers for a solid-fluid mixture. It is interesting to note that these two theories are completely equivalent concerning the constitutive restrictions.

The general results are then applied to the studies of porous media by introducing the concept of porosity. The resulting theory is consistent with Darcy's law and the concepts of pore fluid pressure and effective stress principle in soil mechanics.

**Keywords:** Balance equations, entropy principle, Müller–Liu procedure, volume fraction, pore fluid pressure, effective stress, Darcy's law.

## 1 Introduction

The theories of mixtures in the framework of rational continuum thermodynamics have been developed throughout the sixties and seventies, and subsequent development in various constitutive theories and thermodynamic analysis are too numerous to document (for related references, see [10]). Unlike Irreversible Thermodynamics which postulates a *generalized* Gibbs relation and mandates *linear* constitutive relations in thermodynamic *fluxes/forces*, Rational Thermodynamics relies on the entropy principle to exploit general restrictive conditions on constitutive functions, linear or non-linear. There are basically two different approaches in exploiting the entropy principle. One is the so-called Coleman–Noll procedure, by eliminating the entropy supply with the energy supply of the energy equation. While the other is the so-called Müller–Liu procedure by employing the Lagrange multipliers. Generally speaking, the Coleman–Noll procedure is more restrictive because it requires some specific assumptions concerning the relation between the entropy flux/supply and energy flux/supply. However, it is more widely used owing to its relatively simpler mathematical calculations. For mixture theories, Müller was the first to employ Lagrange multipliers for mixtures of fluids [9] and later many similar theories have been presented mostly with linear constitutive equations to facilitate the tedious calculations and to avoid possible mathematical difficulties.

One of the typical constitutive theory of elastic solid-fluid mixture was presented by Bowen [1], employing Coleman–Noll procedure for large deformation without linear constitutive assumptions. In this paper, we shall employ the Lagrange multipliers in analyzing the entropy principle for such a constitutive theory in order to compare with the general results

obtained by Bowen. In doing so, we refrain from making isotropic and linear constitutive assumptions, but to facilitate mathematical difficulties in dealing with Lagrange multipliers, some simplifying assumptions are made, for one, the Lagrange multiplier for energy equation is identified with the reciprocal of the absolute temperature. It is found that these two theories are completely equivalent concerning the constitutive restrictions.

After a brief review of essential features of the theories of mixtures and the exploitation of constitutive restriction for an elastic solid-fluid mixture with Lagrange multipliers, the general results are applied to the formulation of a theory of porous media by introducing the concept of porosity. The resulting theory is consistent with Darcy's law and the concepts of pore fluid pressure and effective stress principle in soil mechanics.

## 2 Theories of mixtures

We shall briefly review some essential features for continuum framework of mixture theories. We consider a mixture of  $N$  constituents, all of which are supposed to be able to occupy the same region of space simultaneously. Let  $\mathcal{B}_\alpha$  denote the  $\alpha^{\text{th}}$  constituent and  $\kappa_\alpha$  be its reference configuration and denote  $B_\alpha = \kappa_\alpha(\mathcal{B}_\alpha)$ . The motion of  $\mathcal{B}_\alpha$  is a smooth mapping,

$$\chi_\alpha : B_\alpha \times \mathbb{R} \rightarrow \mathbb{E}, \quad \mathbf{x} = \chi_\alpha(\mathbf{X}_\alpha, t), \quad \mathbf{X}_\alpha \in B_\alpha,$$

for each constituent  $\alpha = 1, \dots, N$ . It states that for different motion of each constituent, at the instant  $t$ , there is a material point  $X_\alpha \in \mathcal{B}_\alpha$  in each constituent,  $\mathbf{X}_\alpha = \kappa_\alpha(X_\alpha)$ , that occupies the same spatial position  $\mathbf{x}$  in the Euclidean space  $\mathbb{E}$ . The velocity and the deformation gradient of each constituent are defined as

$$\mathbf{v}_\alpha = \frac{\partial}{\partial t} \chi_\alpha(\mathbf{X}_\alpha, t), \quad \mathbf{F}_\alpha = \nabla_{\mathbf{X}_\alpha} \chi_\alpha(\mathbf{X}_\alpha, t).$$

We introduce the following quantities for the constituent  $\alpha \in \{1, \dots, N\}$ :

- $\rho_\alpha$  mass density of constituent  $\alpha$ .
- $\mathbf{T}_\alpha$  stress tensor of constituent  $\alpha$ .
- $\mathbf{b}_\alpha$  external body force on constituent  $\alpha$ .
- $\varepsilon_\alpha$  internal energy density of constituent  $\alpha$ .
- $\mathbf{q}_\alpha$  energy flux of constituent  $\alpha$ .
- $r_\alpha$  external energy supply of constituent  $\alpha$ .
- $\eta_\alpha$  entropy density of constituent  $\alpha$ .
- $\Phi_\alpha$  entropy flux of constituent  $\alpha$ .
- $s_\alpha$  external entropy supply of constituent  $\alpha$ .
- $\tau_\alpha$  mass production of constituent  $\alpha$ .
- $\mathbf{m}_\alpha$  interaction force on constituent  $\alpha$ .
- $M_\alpha$  interaction moment of momentum on constituent  $\alpha$ .
- $l_\alpha$  energy production of constituent  $\alpha$ .
- $\sigma_\alpha$  entropy production of constituent  $\alpha$ .

### 2.1 Balance laws of each constituent

Following the pioneering work of Truesdell [13], the basic laws of a mixture are given by the following balance equations for mass, linear momentum, moment of momentum, and energy

for each constituent:

$$\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) &= \tau_\alpha, \\
\frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - T_\alpha) - \rho_\alpha \mathbf{b}_\alpha &= \mathbf{m}_\alpha, \\
T_\alpha - T_\alpha^T &= M_\alpha, \\
\frac{\partial}{\partial t} \left( \rho_\alpha \varepsilon_\alpha + \frac{1}{2} \rho_\alpha \mathbf{v}_\alpha^2 \right) + \operatorname{div} \left( \left( \rho_\alpha \varepsilon_\alpha + \frac{1}{2} \rho_\alpha \mathbf{v}_\alpha^2 \right) \mathbf{v}_\alpha + \mathbf{q}_\alpha - T_\alpha^T \mathbf{v}_\alpha \right) \\
&\quad - \rho_\alpha r_\alpha - \rho_\alpha \mathbf{b}_\alpha \cdot \mathbf{v}_\alpha = l_\alpha,
\end{aligned} \tag{2.1}$$

If the mixture reduces to a single constituent, the right hand side of the above equations are zero, and we recover the balance laws for a single body. The terms on the right hand side thus represent the physical transfers among different constituents, such as chemical reactions, interaction forces and energy transfer. In a mixture as a whole, we assume that such physical transfers are solely due to exchanges among constituents. Therefore, we postulate that the mixture as a whole should behave like a single body. This is expressed by the following relations:

$$\sum_\alpha \tau_\alpha = 0, \quad \sum_\alpha \mathbf{m}_\alpha = 0, \quad \sum_\alpha M_\alpha = 0, \quad \sum_\alpha l_\alpha = 0. \tag{2.2}$$

## 2.2 Balance laws of the mixture

The behavior of a mixture as a whole will be assumed to be governed by the same equations of balance as a single body. Indeed, by summing up the balance equations (2.1) over all constituents, we obtain the usual balance equations of mass, linear momentum, and energy for the mixture,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - T) &= \rho \mathbf{b}, \\
\frac{\partial}{\partial t} \left( \rho \varepsilon + \frac{1}{2} \rho \mathbf{v}^2 \right) + \operatorname{div} \left( \left( \rho \varepsilon + \frac{1}{2} \rho \mathbf{v}^2 \right) \mathbf{v} + \mathbf{q} - T^T \mathbf{v} \right) &= \rho r + \rho \mathbf{b} \cdot \mathbf{v},
\end{aligned} \tag{2.3}$$

by the use of the relations (2.2) and the following definition of the corresponding quantities for the mixture,

$$\begin{aligned}
\rho &= \sum_\alpha \rho_\alpha, & \mathbf{v} &= \sum_\alpha \frac{\rho_\alpha}{\rho} \mathbf{v}_\alpha, \\
T &= \sum_\alpha (T_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha), & \mathbf{b} &= \sum_\alpha \frac{\rho_\alpha}{\rho} \mathbf{b}_\alpha,
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\varepsilon &= \sum_\alpha \frac{\rho_\alpha}{\rho} \left( \varepsilon_\alpha + \frac{1}{2} \mathbf{u}_\alpha^2 \right), & r &= \sum_\alpha \frac{\rho_\alpha}{\rho} (r_\alpha + \mathbf{b}_\alpha \cdot \mathbf{u}_\alpha), \\
\mathbf{q} &= \sum_\alpha \left( \mathbf{q}_\alpha + \rho_\alpha \left( \varepsilon_\alpha + \frac{1}{2} \mathbf{u}_\alpha^2 \right) \mathbf{u}_\alpha - T_\alpha^T \mathbf{u}_\alpha \right),
\end{aligned} \tag{2.5}$$

Note that in the relation (2.4) and (2.5), we have introduced the diffusive velocity of constituent  $\alpha$  relative to the mixture,

$$\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}.$$

Also note that  $\sum M_\alpha = 0$  implies the symmetry of  $\sum T_\alpha$  as well as the total stress  $T$ , while the partial stress  $T_\alpha$  is not symmetric in general.

## 2.3 The entropy inequality

The balance of entropy of constituent  $\alpha$  can be written as

$$\frac{\partial \rho_\alpha \eta_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \eta_\alpha \mathbf{v}_\alpha + \boldsymbol{\Phi}_\alpha) - \rho_\alpha s_\alpha = \sigma_\alpha.$$

Since entropy is not a conservative quantity, we shall allow the partial production of entropy of a constituent be of any amount provided that the total production of entropy for the mixture be non-negative,

$$\sum_\alpha \sigma_\alpha \geq 0. \quad (2.6)$$

By summing up the equations of all constituents  $\alpha \in \{1, \dots, N\}$  and introducing the corresponding quantities for the mixture,

$$\begin{aligned} \eta &= \sum_\alpha \frac{\rho_\alpha}{\rho} \eta_\alpha, & s &= \sum_\alpha \frac{\rho_\alpha}{\rho} s_\alpha, \\ \boldsymbol{\Phi} &= \sum_\alpha (\boldsymbol{\Phi}_\alpha + \rho_\alpha \eta_\alpha \mathbf{u}_\alpha), \end{aligned} \quad (2.7)$$

we obtain the usual entropy inequality for the mixture as a single body,

$$\frac{\partial \rho \eta}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v} + \boldsymbol{\Phi}) - \rho s \geq 0. \quad (2.8)$$

Note that we do not postulate a similar entropy inequality for individual constituent. The partial entropy production could be negative provided that the total production of the mixture is non-negative, otherwise it will lead to very restrictive physical results.

The use of the general entropy inequality (2.8) for the exploitation of its restrictions on constitutive models is often quite sophisticated and could be rather tedious. In application, assumption for the specific form of the entropy flux and entropy supply for the constituents is usually adopted, namely,

$$\boldsymbol{\Phi}_\alpha = \frac{1}{\theta_\alpha} \mathbf{q}_\alpha, \quad s_\alpha = \frac{1}{\theta_\alpha} r_\alpha, \quad (2.9)$$

where  $\theta_\alpha$  is the temperature of the constituent  $\alpha$ . Under this assumption, from (2.7), we have

$$\boldsymbol{\Phi} = \sum_\alpha \left( \frac{\mathbf{q}_\alpha}{\theta_\alpha} + \rho_\alpha \eta_\alpha \mathbf{u}_\alpha \right), \quad s = \sum_\alpha \frac{\rho_\alpha}{\rho} \frac{r_\alpha}{\theta_\alpha}.$$

For further discussions, we shall assume that energy exchange among constituents is quick enough relative to characteristic observation time, so that all the constituents have the same common temperature. We can then rewrite the above relations as

$$\begin{aligned} \boldsymbol{\Phi} &= \frac{1}{\theta} \left( \mathbf{q} - \sum_\alpha (\rho_\alpha (K_\alpha + \frac{1}{2} \mathbf{u}_\alpha^2) I) \mathbf{u}_\alpha \right), \\ s &= \frac{1}{\theta} \left( r - \sum_\alpha \frac{\rho_\alpha}{\rho} \mathbf{u}_\alpha \cdot \mathbf{b}_\alpha \right), \end{aligned} \quad (2.10)$$

where

$$K_\alpha = \psi_\alpha I - \frac{T_\alpha^T}{\rho_\alpha}, \quad \psi_\alpha = \varepsilon_\alpha - \theta \eta_\alpha, \quad (2.11)$$

will be called the chemical potential tensor and the free energy of constituent  $\alpha$  respectively.

Even though we have adopted the Clausius–Duhem assumptions (2.9) for each constituent, for the mixture as a single body, from (2.10) such assumptions are not valid. In other words, the Clausius–Duhem assumptions  $\boldsymbol{\Phi} = \mathbf{q}/\theta$  and  $s = r/\theta$  may not be appropriate for thermodynamic considerations of material bodies in general.

## 2.4 Jump conditions at semi-permeable surface

Let  $\mathcal{S}$  be a surface within the mixture and  $[A]$  be the jump of a quantity  $A$  across  $\mathcal{S}$ , defined by

$$[A] = A^+ - A^-.$$

We assume that the one-side limits  $A^+$  and  $A^-$  exist and the jump is smooth on  $\mathcal{S}$ . Such a surface is called a singular surface. Let  $\mathbf{u}^*$  and  $\mathbf{n}$  denote the velocity and the unit normal of  $\mathcal{S}$  respectively.

In addition to the balance laws (2.3) and (2.8) at a regular point where relevant fields are continuous, at a singular point, i.e., a point on  $\mathcal{S}$ , the balance laws for the mixture take the form of the following jump conditions of mass, momentum, energy and entropy:

$$\begin{aligned} [\rho(\mathbf{v} - \mathbf{u}^*)] \cdot \mathbf{n} &= 0, \\ [\rho\mathbf{v} \otimes (\mathbf{v} - \mathbf{u}^*) - T]\mathbf{n} &= 0, \\ [\mathbf{q} - T\mathbf{v} + \rho(\varepsilon + \frac{1}{2}\mathbf{v}^2)(\mathbf{v} - \mathbf{u}^*)] \cdot \mathbf{n} &= 0, \\ [\Phi + \rho\eta(\mathbf{v} - \mathbf{u}^*)] \cdot \mathbf{n} &\geq 0. \end{aligned} \tag{2.12}$$

We do not postulate the similar jump conditions for each constituent, because it is generally too restrictive and may even lead to unrealistic results<sup>1</sup>.

A singular surface will be called ideal, if across it the temperature is continuous,  $[\theta] = 0$ . We assume further that across an ideal singular surface, the entropy jump is null,

$$[\Phi + \rho\eta(\mathbf{v} - \mathbf{u}^*)] \cdot \mathbf{n} = 0, \tag{2.13}$$

and the entropy flux  $\Phi$  is given by the relation (2.10)<sub>1</sub>.

The diffusive velocity of constituent  $\alpha$  relative to the surface  $\mathcal{S}$  is defined as

$$V_\alpha = \mathbf{v}_\alpha - \mathbf{u}^*.$$

If  $V_\alpha = 0$ , we say that  $\mathcal{S}$  is not permeable to the constituent  $\alpha$ , i.e., the constituent does not go across the surface. A singular surface which is not permeable to all constituents except the constituent  $\nu$  is called a semi-permeable surface with respect to the constituent  $\nu$ , or simply a  $\nu$ -permeable surface. We have the following result (see [4]):

**Theorem.** *Let  $\mathcal{S}$  be an ideal  $\nu$ -permeable surface, then by the use of the jump conditions (2.12)<sub>1,2</sub> and (2.13), the energy jump condition (2.12)<sub>3</sub> becomes*

$$[(K_\nu + \frac{1}{2}V_\nu^2 I)\rho_\nu V_\nu] \cdot \mathbf{n} = 0, \tag{2.14}$$

where  $K_\nu$  is the chemical potential tensor of the constituent  $\nu$ .

## 3 An elastic solid-fluid mixture

We consider a non-reacting binary elastic solid-fluid mixture by the constitutive equation of the form:

$$f = \mathcal{F}(\theta, F_s, F_f, \text{grad } \theta, \text{grad } F_s, \text{grad } F_f, \mathbf{v}_s, \mathbf{v}_f), \tag{3.1}$$

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<sup>1</sup>It is shown in the end remark of [4], that the jump conditions for each constituent cannot be valid in the case of porous media

where  $\theta$  is the temperature,  $F_s$  and  $F_f$  are the deformation gradients of the solid and fluid constituents respectively, with the following balance equations (from (2.1)<sub>1,2</sub> and (2.3)):

$$\begin{aligned}
\dot{\rho}_s + \rho_s \operatorname{div} \mathbf{v}_s &= 0, \\
\dot{\rho}_f + \rho_f \operatorname{div} \mathbf{v}_f &= 0, \\
\rho_s \dot{\mathbf{v}}_s - \operatorname{div} T_s + \mathbf{m}_f &= \rho_s \mathbf{b}_s, \\
\rho_f \dot{\mathbf{v}}_f - \operatorname{div} T_f - \mathbf{m}_f &= \rho_f \mathbf{b}_f, \\
\rho \dot{\varepsilon} + \operatorname{div} \mathbf{q} - \operatorname{tr}(T \operatorname{grad} \mathbf{v}) &= \rho r,
\end{aligned} \tag{3.2}$$

where the material derivatives with respect to the constituent and the mixture are used,

$$\begin{aligned}
\dot{y}_\alpha &= \frac{\partial y_\alpha}{\partial t} + (\operatorname{grad} y_\alpha) \mathbf{v}_\alpha, \\
\dot{y} &= \frac{\partial y}{\partial t} + (\operatorname{grad} y) \mathbf{v}.
\end{aligned}$$

Moreover, for given density  $\bar{\rho}_s$  in the reference configuration, the partial balance equation (3.2)<sub>1</sub> for the solid constituent can be integrated to

$$\rho_s = \frac{\bar{\rho}_s}{\det F_s}. \tag{3.3}$$

Note that since only a common temperature is considered for all constituents, only the energy equation of the mixture, instead of the partial energy equations (2.1)<sub>4</sub> of all constituents, is needed in the system of governing equations (3.2). Moreover, since the partial stresses are not assumed to be symmetric, the equation of partial moment of momentum (2.1)<sub>3</sub> is not explicitly needed in (3.2). In other words, the interaction moment  $M_\alpha$  can simply be determined from the skew-symmetric part of the partial stress  $T_\alpha$ .

To establish field equations of the basic field variables,  $\{\rho_s, \rho_f, \chi_s, \chi_f, \theta\}$ , constitutive equations for the quantities in the balance equations,

$$f = \{T_s, T_f, \varepsilon, \mathbf{q}, \mathbf{m}_f\} \tag{3.4}$$

of the material model must be specified. The main objective of constitutive theory is to derive restrictions on constitutive functions based on some physical principles. First of all, according to the principle of material frame-indifference, since the above constitutive quantities are objective tensor, scalar and vector quantities, an immediate consequence follows, namely, constitutive functions can not depend on partial velocities independently, rather must depend on the relative velocity, say,  $V = \mathbf{v}_f - \mathbf{v}_s$ . Secondly, the material symmetry condition for fluid constituent requires the dependence on the deformation gradient be reduced to the dependence of the determinant of the deformation gradient only. Therefore, we may redefine the constitutive functions (3.1) in terms of the following state variables,

$$f = \mathcal{F}(\theta, \rho_f, F_s, \operatorname{grad} \theta, \operatorname{grad} \rho_f, \operatorname{grad} F_s, V). \tag{3.5}$$

For given fields of external supplies  $\{\mathbf{b}_s(\mathbf{x}, t), \mathbf{b}_f(\mathbf{x}, t), r(\mathbf{x}, t)\}$ , any solution of the system of field equations,  $\{\rho_f(\mathbf{x}, t), \mathbf{v}_f(\mathbf{x}, t), \mathbf{v}_s(\mathbf{x}, t), F_s(\mathbf{x}, t), \theta(\mathbf{x}, t)\}$ , will be called a thermodynamic process.

### Some useful identities

For convenience, we define the material time derivative  $\overset{\star}{y}$  for any function  $y$ , and the relative velocity  $V$  with respect to the solid constituent,

$$\overset{\star}{y} = \frac{\partial y}{\partial t} + (\operatorname{grad} y) \mathbf{v}_s, \quad V = \mathbf{v}_f - \mathbf{v}_s.$$

From these definitions, we have the following identities:

$$\begin{aligned}\mathbf{u}_f &= \mathbf{v}_f - \mathbf{v} = \frac{\rho_s}{\rho} V, \\ \mathbf{u}_s &= \mathbf{v}_s - \mathbf{v} = -\frac{\rho_f}{\rho} V,\end{aligned}$$

since  $\rho \mathbf{v} = \rho_s \mathbf{v}_s + \rho_f \mathbf{v}_f$ , and

$$\begin{aligned}\dot{y}_s &= \dot{y}_s^*, \\ \dot{y}_f &= \dot{y}_f^* + (\text{grad } y_f) V, \\ \dot{y} &= \dot{y}^* + \frac{\rho_f}{\rho} (\text{grad } y) V,\end{aligned}$$

and

$$\text{grad } \mathbf{v} = \text{grad } \mathbf{v}_s + \frac{\rho_f}{\rho} \text{grad } V + V \otimes \text{grad} \left( \frac{\rho_f}{\rho} \right).$$

These identities will be frequently used later without further reminder. From (2.4) and (2.5), we also have

$$\begin{aligned}T &= T_I - \frac{\rho_s \rho_f}{\rho} V \otimes V, \\ \varepsilon &= \varepsilon_I + \frac{1}{2} \frac{\rho_s \rho_f}{\rho^2} V^2,\end{aligned}\tag{3.6}$$

where  $T_I$  and  $\varepsilon_I$  stand for the internal parts without the explicit dependence on the relative velocity,

$$T_I = T_s + T_f, \quad \varepsilon_I = \frac{\rho_s}{\rho} \varepsilon_s + \frac{\rho_f}{\rho} \varepsilon_f.$$

Another well-known identity,  $\text{grad } \mathbf{v} = \dot{F} F^{-1}$ , for the solid constituent gives

$$\text{grad } \mathbf{v}_s = \dot{F}_s^* F_s^{-1}.\tag{3.7}$$

Note that from the material time derivative of the relation (3.3),  $\dot{\rho}_s^* = -\rho_s \text{tr}(\dot{F}_s^* F_s^{-1})$ , the trace of the relation (3.7) reduces to the mass balance of the solid constituent (3.2)<sub>1</sub>. In the evaluation of entropy principle, this tensor equation (3.7) will be taken into account replacing the scalar equation (3.2)<sub>1</sub>.

### 3.1 Entropy principle

The essential task of constitutive theory lies in the evaluation of the entropy principle, which requires that the entropy inequality (from (2.8)),

$$\rho \dot{\eta} + \text{div } \Phi - \rho s \geq 0,\tag{3.8}$$

be satisfied for every thermodynamic process. The entropy density and the entropy flux will also be given by constitutive equation of the form (3.5).

Since material properties should not depend on external supplies which characterizes the environment the material bodies encountered, therefore, one can consider supply-free bodies for the evaluation of constitutive restrictions from the entropy principle, by assuming that there is no external entropy supply when there are no any external body force and energy supplies on the body. In this case, according to Liu [3], the requirement that the entropy inequality (3.8) must hold for any thermodynamic process can be analyzed with the use of Lagrange multipliers.

**Method of Lagrange multipliers.** *There exist Lagrange multipliers  $\Lambda^{L_s}$ ,  $\Lambda^{\rho_f}$ ,  $\Lambda^{v_s}$ ,  $\Lambda^{v_f}$ , and  $\Lambda^\varepsilon$ , depending on the state variables, such that the inequality*

$$\begin{aligned}
& \rho \dot{\eta} + \rho_f \text{grad } \eta \cdot V + \text{div } \Phi \\
& - \Lambda^{L_s} \cdot \rho_s (-\dot{F}_s F_s^{-1} + \text{grad } \mathbf{v}_s) \\
& - \Lambda^{\rho_f} (\dot{\rho}_f + \text{grad } \rho_f \cdot V + \rho_f \text{div } \mathbf{v}_f) \\
& - \Lambda^{v_s} \cdot (\rho_s \dot{\mathbf{v}}_s - \text{div } T_s + \mathbf{m}_f) \\
& - \Lambda^{v_f} \cdot (\rho_f \dot{\mathbf{v}}_f + \rho_f (\text{grad } v_f) V - \text{div } T_f - \mathbf{m}_f) \\
& - \Lambda^\varepsilon (\rho \dot{\varepsilon} + \rho_f \text{grad } \varepsilon \cdot V + \text{div } \mathbf{q} - \text{tr}(T \text{grad } \mathbf{v})) \geq 0,
\end{aligned} \tag{3.9}$$

must hold for any analytic fields  $\{\rho_f(\mathbf{x}, t), \mathbf{v}_f(\mathbf{x}, t), \mathbf{v}_s(\mathbf{x}, t), F_s(\mathbf{x}, t), \theta(\mathbf{x}, t)\}$ .<sup>2</sup>

Note that after introducing the constitutive relations into (3.9), the inequality assumes the following form:

$$\sum_a S_a \cdot X_a + \sigma \geq 0, \tag{3.10}$$

where  $X_a = (\dot{F}_s, \dot{\rho}_f, \dot{V}, \dot{\mathbf{v}}_s, \dot{\theta}, (\nabla \theta)^*, (\nabla \rho_f)^*, (\nabla F_s)^*; \nabla V, \nabla \mathbf{v}_s, \nabla(\nabla \theta), \nabla(\nabla \rho_f), \nabla(\nabla F_s))$ , while  $S_a$  and  $\sigma$  are functions of state variables. We have written the grad operator as  $\nabla$ .

Note that the inequality (3.10) is linear in  $X_a$ , and the values of  $X_a$  can be given independently of the values of  $S_a$  and  $\sigma$ . This implies that  $S_a$  (respecting the part involved with the symmetry of the second gradients in the corresponding  $X_a$ ) must vanish, otherwise, it is possible to choose some values of  $X_a$  such that the inequality is violated. By introducing the notations,

$$\begin{aligned}
H_X &= \frac{\partial \eta}{\partial X} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial X}, \\
M_X &= \frac{\partial \Phi}{\partial X} - \Lambda^\varepsilon \frac{\partial \mathbf{q}}{\partial X} + \Lambda^{v_s} \cdot \frac{\partial T_s}{\partial X} + \Lambda^{v_f} \cdot \frac{\partial T_f}{\partial X} + V \otimes \rho_f H_X,
\end{aligned} \tag{3.11}$$

we have the following entropy density conditions:

$$\begin{aligned}
\rho H_{F_s} + \rho_s \Lambda^{L_s} F_s^{-T} &= 0, & \rho H_{\rho_f} - \Lambda^{\rho_f} &= 0, \\
\rho H_V - \rho_f \Lambda^{v_f} &= 0, & -\rho_s \Lambda^{v_s} - \rho_f \Lambda^{v_f} &= 0,
\end{aligned} \tag{3.12}$$

and

$$H_\theta = 0, \quad H_{\nabla \theta} = 0, \quad H_{\nabla \rho_f} = 0, \quad H_{\nabla F_s} = 0. \tag{3.13}$$

Moreover, we have the entropy flux conditions:

$$\begin{aligned}
M_V + \Lambda^\varepsilon \frac{\rho_f}{\rho} T - \Lambda^{\rho_f} \rho_f I - V \otimes \rho_f \Lambda^{v_f} &= 0, \\
\Lambda^\varepsilon T - \rho_s \Lambda^{L_s} - \rho_f \Lambda^{\rho_f} I - V \otimes \rho_f \Lambda^{v_f} &= 0,
\end{aligned} \tag{3.14}$$

and

$$(M_{\nabla \theta})_{\text{sym}} = 0, \quad (M_{\nabla \rho_f})_{\text{sym}} = 0, \quad (M_{\nabla F_s})_{\text{sym}} = 0, \tag{3.15}$$

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<sup>2</sup>For general information on the applicability of the method of Lagrange multipliers, please refer to [3] or Chap. 7 of [6].



where  $(\bullet)_{\text{sym}}$  stand for the symmetric part of the expression resulting from the symmetry of the second gradient in  $X_a$  in (3.10). There is a remaining entropy production inequality,

$$\begin{aligned} \sigma &= M_\theta \cdot \nabla \theta + M_{\rho_f} \cdot \nabla \rho_f + M_{F_s} \cdot \nabla F_s \\ &+ A^\varepsilon T \cdot \left( V \otimes \nabla \frac{\rho_f}{\rho} \right) - A^{\rho_f} V \cdot \nabla \rho_f - (A^{v_s} - A^{v_f}) \cdot \mathbf{m}_f \geq 0. \end{aligned} \quad (3.16)$$

The relations concerning the entropy density and the entropy flux from (3.12) through (3.16) are the complete set of consequences from the entropy principle. It remains to analyze the restrictions on the constitutive functions from these relations and the physical significance of the Lagrange multipliers.

### 3.2 Further evaluations

We shall remark that, even in the case of a single elastic solid body, a rigorous evaluation of the entropy flux relation with the Lagrange multiplier remains generally impossible without some further assumptions, such as material isotropy or some explicit representations of constitutive functions (see [2, 7, 8, 9]). To facilitate further evaluations and avoid technical mathematical difficulties, we shall also make some simplifying assumptions consistent with the general description of the classical mixture theory given in the previous section, but refrain from making linear and isotropic constitutive assumptions.

#### Entropy density relation

The entropy relations (3.12) and (3.13) can be summarized in the following differential form,

$$d\eta = A^\varepsilon d\varepsilon + \frac{\rho_f}{\rho} A^{v_f} \cdot dV + \frac{1}{\rho} A^{\rho_f} d\rho_f - \frac{\rho_s}{\rho} A^{L_s} F_s^{-T} \cdot dF_s,$$

which by the use of (3.6)<sub>2</sub> becomes

$$\begin{aligned} d\eta &= A^\varepsilon d\varepsilon_1 + \left( A^\varepsilon \frac{\rho_s \rho_f}{\rho^2} V + \frac{\rho_f}{\rho} A^{v_f} \right) \cdot dV \\ &+ \left( A^\varepsilon \frac{1}{2} \frac{\rho_s}{\rho^3} (\rho_s - \rho_f) V^2 + \frac{1}{\rho} A^{\rho_f} \right) d\rho_f \\ &- \left( A^\varepsilon \frac{1}{2} \frac{\rho_f}{\rho^3} (\rho_f - \rho_s) V^2 I + \frac{1}{\rho} A^{L_s} \right) \rho_s F_s^{-T} \cdot dF_s. \end{aligned} \quad (3.17)$$

This resembles a generalized Gibbs relation, which suggests that for simplicity, one could identify the Lagrange multiplier  $A^\varepsilon$  as the reciprocal of the absolute temperature  $\theta$ ,

$$A^\varepsilon = \frac{1}{\theta}, \quad (3.18)$$

and take (see [9] and [2] Sec. 7.6)

$$\begin{aligned} A^{v_f} &= -A^\varepsilon \frac{\rho_s}{\rho} V = -\frac{1}{\theta} \mathbf{u}_f, \\ A^{v_s} &= A^\varepsilon \frac{\rho_f}{\rho} V = -\frac{1}{\theta} \mathbf{u}_s. \end{aligned} \quad (3.19)$$

Note that the second relation follows from the first relation and (3.12)<sub>4</sub> and is consistent with the identity  $\rho_s \mathbf{u}_s + \rho_f \mathbf{u}_f = 0$ .

With the assumptions (3.18) and (3.19)<sub>1</sub>, the relation (3.17) reduces to

$$d\eta = \frac{1}{\theta} d\varepsilon_1 + \frac{1}{\rho} A_1^{\rho_f} d\rho_f - \frac{\rho_s}{\rho} A_1^{L_s} F_s^{-T} \cdot dF_s, \quad (3.20)$$

where we have defined the internal parts without explicit  $V$ -dependent Lagrange multipliers as

$$\begin{aligned}\Lambda_I^{\rho_f} &= \Lambda^{\rho_f} + \Lambda^\varepsilon \frac{1}{2} \frac{\rho_s}{\rho^2} (\rho_s - \rho_f) (V \cdot V), \\ \Lambda_I^{L_s} &= \Lambda^{L_s} + \Lambda^\varepsilon \frac{1}{2} \frac{\rho_f}{\rho^2} (\rho_f - \rho_s) (V \cdot V) I.\end{aligned}\tag{3.21}$$

The relation (3.20) can also be rewritten as

$$d\psi_I = -\eta d\theta - \frac{\theta}{\rho} \left( \Lambda_I^{\rho_f} d\rho_f - \rho_s \Lambda_I^{L_s} F_s^{-T} \cdot dF_s \right),\tag{3.22}$$

or

$$d(\rho\psi_I) = -\rho\eta d\theta + \left( \psi_I - \theta \Lambda_I^{\rho_f} \right) d\rho_f - \left( \psi_I I - \theta \Lambda_I^{L_s} \right) \rho_s F_s^{-T} \cdot dF_s,$$

where

$$\psi_I = \varepsilon_I - \theta\eta = \frac{\rho_s}{\rho} \psi_s + \frac{\rho_f}{\rho} \psi_f,$$

is the internal part of the free energy density. It follows from this Gibbs relation that

$$\begin{aligned}\psi_I &= \psi_I(\theta, \rho_f, F_s), \quad \eta = -\frac{\partial\psi_I}{\partial\theta}, \\ \Lambda_I^{\rho_f} &= -\frac{1}{\theta} \rho \frac{\partial\psi_I}{\partial\rho_f} = \frac{1}{\theta} \left( \psi_I - \frac{\partial\rho\psi_I}{\partial\rho_f} \right), \\ \Lambda_I^{L_s} &= \frac{1}{\theta} \frac{\rho}{\rho_s} \frac{\partial\psi_I}{\partial F_s} F_s^T = \frac{1}{\theta} \left( \psi_I I + \frac{1}{\rho_s} \frac{\partial\rho\psi_I}{\partial F_s} F_s^T \right).\end{aligned}\tag{3.23}$$

### Entropy flux relation

The second part of evaluation concerns the entropy flux relations (3.14) and (3.15). This is in fact the most challenging and rewarding part of the evaluation procedure with Lagrange multipliers (see [5]). For example, in [7] for isotropic viscoelastic bodies, the mathematically complicated procedure leads to the entropy flux–heat flux relation and the proof that the Lagrange multiplier  $\Lambda^\varepsilon$  is a function of the temperature only (which for simplicity, we have taken as an assumption in (3.18)). In the present case, a similar analysis remains a very challenging mathematical problem in general.

Meanwhile, we shall cast the relation (3.14)<sub>1</sub>,

$$\begin{aligned}\frac{\partial\Phi}{\partial V} - \frac{1}{\theta} \frac{\partial\mathbf{q}}{\partial V} &= - \left( \Lambda^{v_s} \cdot \frac{\partial T_s}{\partial V} + \Lambda^{v_f} \cdot \frac{\partial T_f}{\partial V} + \frac{1}{\theta} \frac{\rho_f}{\rho} T - V \otimes \rho_f \Lambda^{v_f} \right) \\ &\quad - V \otimes \rho_f \left( \frac{\partial\eta}{\partial V} - \frac{1}{\theta} \frac{\partial\varepsilon}{\partial V} \right) + \Lambda^{\rho_f} \rho_f I.\end{aligned}$$

into a more convenient form.

By the use of (3.19), (3.21)<sub>1</sub>, (3.23)<sub>3</sub>, and (3.6), with the following intermediate results:

$$\begin{aligned}\Lambda^{v_s} \cdot \frac{\partial T_s}{\partial V} + \Lambda^{v_f} \cdot \frac{\partial T_f}{\partial V} + \frac{1}{\theta} \frac{\rho_f}{\rho} T - V \otimes \rho_f \Lambda^{v_f} \\ = \frac{1}{\theta} \frac{\rho_s \rho_f}{\rho} \left\{ V \cdot \left( \frac{1}{\rho_s} \frac{\partial T_s}{\partial V} - \frac{1}{\rho_f} \frac{\partial T_f}{\partial V} \right) + \left( \frac{T_s^T}{\rho_s} - \frac{T_f^T}{\rho_f} \right) + \frac{\rho_s}{\rho} V \otimes V \right\} + \frac{1}{\theta} T_f^T.\end{aligned}$$

and

$$\begin{aligned}V \otimes \rho_f \left( \frac{\partial\eta}{\partial V} - \frac{1}{\theta} \frac{\partial\varepsilon}{\partial V} \right) &= -\frac{1}{\theta} \left( V \otimes \rho_f \frac{\partial\psi_I}{\partial V} + \frac{\rho_s \rho_f^2}{\rho^2} V \otimes V \right) \\ &= -\frac{1}{\theta} \left\{ \left( \rho_f (\psi_f - \psi_I) I + V \otimes \rho_f \frac{\partial\psi_f}{\partial V} - \frac{\partial}{\partial V} (\rho_f (\psi_f - \psi_I) V) \right) + \frac{\rho_s \rho_f^2}{\rho^2} V \otimes V \right\},\end{aligned}$$

we arrive at the relation,

$$\frac{\partial}{\partial V} \left( \Phi - \frac{1}{\theta} (\mathbf{q} + \mathbf{k}) \right) = -\frac{1}{\theta} \left( T_f^T - \rho_f \psi_f I + \rho_f \frac{\partial \rho \psi_f}{\partial \rho_f} I - \rho_f V \otimes \frac{\partial \psi_f}{\partial V} \right). \quad (3.24)$$

In this relation, we have introduced the definition of an extra flux,

$$\mathbf{k} = -\rho_f (\psi_f - \psi_I) V - \frac{\rho_s \rho_f}{\rho} \left( \frac{T_s^T}{\rho_s} - \frac{T_f^T}{\rho_f} \right) V - \frac{1}{2} \frac{\rho_s \rho_f}{\rho^2} (\rho_s - \rho_f) (V \cdot V) V, \quad (3.25)$$

which by the use of (3.19), can be written as

$$\mathbf{k} = - \sum_{\alpha=s,f} \left( \rho_\alpha \psi_\alpha I - T_\alpha^T + \frac{1}{2} \rho_\alpha (\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) I \right) \mathbf{u}_\alpha.$$

On the other hand, with the extra flux  $\mathbf{k}$ , the entropy flux relation (2.10)<sub>1</sub>, given in the previous section, becomes

$$\Phi = \frac{1}{\theta} (\mathbf{q} + \mathbf{k}). \quad (3.26)$$

To proceed with our evaluation, without attempting a rigorous proof, we shall make another simplifying assumption that this entropy flux relation (3.26) holds. An immediate consequence follows from (3.24) that the right hand side should vanish, which gives

$$\begin{aligned} T_f &= \rho_f \psi_f I - \frac{\partial \rho \psi_f}{\partial \rho_f} \rho_f I + \rho_f \frac{\partial \psi_f}{\partial V} \otimes V, \\ T_s &= \rho_s \psi_s I + \frac{\partial \rho \psi_s}{\partial F_s} F_s^T + \rho_s \frac{\partial \psi_s}{\partial V} \otimes V. \end{aligned} \quad (3.27)$$

The second relation follows from the condition (3.14)<sub>2</sub> by the use of (3.6)<sub>1</sub> and (3.21).

Furthermore, with (3.25), (3.26) and (3.23)<sub>1</sub>, the definition (3.11) gives rise to

$$M_X = -\frac{1}{\theta} V \otimes \rho_f \frac{\partial \psi_f}{\partial X} \quad \text{for} \quad X = \{\nabla \theta, \nabla \rho_f, \nabla F_s\},$$

and the flux relations (3.15) imply that the symmetric part must vanish,

$$V \otimes \frac{\partial \psi_f}{\partial X} + \frac{\partial \psi_f}{\partial X} \otimes V = 0,$$

which in turns implies  $\partial \psi_f / \partial X = 0$ . To see this, consider  $A \otimes B + B \otimes A = 0$ , then

$$\begin{aligned} A \cdot (A \otimes B + B \otimes A) \cdot B &= 0, \\ \iff (A \cdot A)(B \cdot B) + (A \cdot B)^2 &= 0, \\ \iff A = 0 \quad \text{and/or} \quad B = 0. \end{aligned}$$

Therefore, we have proved that

$$\psi_f = \psi_f(\theta, \rho_f, F_s, V), \quad \psi_s = \psi_s(\theta, \rho_f, F_s, V). \quad (3.28)$$

### Entropy production inequality

With previous results, after a long and straightforward calculation, the remaining entropy production inequality (3.16) reduces to

$$\begin{aligned} \sigma(\nabla\theta, \nabla\rho_f, \nabla F_s, V) &= \left\{ -\frac{1}{\theta^2} (\mathbf{q} + \mathbf{k}) - \frac{1}{\theta} \rho_f \left( \frac{\partial\psi_f}{\partial\theta} + \eta \right) V \right\} \cdot \nabla\theta \\ &+ \frac{1}{\theta} \frac{\partial\rho_s\psi_s}{\partial\rho_f} V \cdot \nabla\rho_f - \frac{1}{\theta} \frac{\partial\rho_f\psi_f}{\partial F_s} V \cdot \nabla F_s - \frac{1}{\theta} V \cdot \mathbf{m}_f + o(|V|^3) \geq 0, \end{aligned} \quad (3.29)$$

where  $o(|V|^3)$  stands for the terms containing  $(V \cdot V)V$  from the velocity-dependent parts of the internal energy  $\varepsilon$  and the stress tensor  $T$ , which are irrelevant for further evaluation.

Note that  $\sigma(0, \nabla\rho_f, \nabla F_s, 0) = 0$ , therefore, we can define an equilibrium process, namely without entropy production, as a process with  $\nabla\theta = 0$  and  $V = 0$ . From (3.29), it states that the entropy production density  $\sigma(\nabla\theta, \nabla\rho_f, \nabla F_s, V)$  attains its minimum at an equilibrium process, which is necessary that the following conditions be satisfied:

$$\left. \frac{\partial\sigma}{\partial X_A} \right|_E = 0, \quad \left. \frac{\partial^2\sigma}{\partial X_A \partial X_B} \right|_E \geq 0, \quad \text{for } X_A = \{\nabla\theta, V\}.$$

In particular, for  $\left. \frac{\partial\sigma}{\partial V} \right|_E = 0$  and  $\left. \frac{\partial\sigma}{\partial\nabla\theta} \right|_E = 0$ , we obtain

$$\mathbf{m}_f^0 = \frac{\partial\rho_s\psi_s^0}{\partial\rho_f} \text{grad } \rho_f - \frac{\partial\rho_f\psi_f^0}{\partial F_s} \cdot \text{grad } F_s, \quad \mathbf{q}^0 = 0. \quad (3.30)$$

where the superscript 0 indicates the value in equilibrium.

### Summary of results

Thermodynamic considerations of such a mixture theory has been considered by Bowen ([1], see also [13] Appendix 5A) in which consequences of the entropy principle have been analyzed with Coleman-Noll procedure based on the assumptions (2.10)<sub>1,2</sub>. It is quite interesting and perhaps even expected that the present analysis (with Müller-Liu procedure, as sometimes referred to) would give rise to exactly the same results, which can be summarized in the following constitutive equations ((3.4) and (3.5)):

$$\begin{aligned} \psi_I &= \psi_I(\theta, \rho_f, F_s), & \varepsilon_I &= \psi_I - \theta \frac{\partial\psi_I}{\partial\theta}, \\ \psi_f &= \psi_f(\theta, \rho_f, F_s, V), & \psi_s &= \psi_s(\theta, \rho_f, F_s, V), \\ T_f &= \rho_f \psi_f I - \frac{\partial\rho\psi_I}{\partial\rho_f} \rho_f I + \rho_f \frac{\partial\psi_f}{\partial V} \otimes V, \\ T_s &= \rho_s \psi_s I + \frac{\partial\rho\psi_I}{\partial F_s} F_s^T + \rho_s \frac{\partial\psi_s}{\partial V} \otimes V, \end{aligned} \quad (3.31)$$

and in equilibrium,

$$\mathbf{m}_f^0 = \frac{\partial\rho_s\psi_s^0}{\partial\rho_f} \text{grad } \rho_f - \frac{\partial\rho_f\psi_f^0}{\partial F_s} \cdot \text{grad } F_s, \quad \mathbf{q}^0 = 0. \quad (3.32)$$

### 3.3 Chemical potential at fluid-permeable surface

If we define the equilibrium chemical potential of the fluid and the equilibrium partial fluid pressure as

$$\mu_f = \frac{\partial\rho\psi_I}{\partial\rho_f}, \quad p_f = \rho_f(\mu_f - \psi_f^0), \quad (3.33)$$

then the equilibrium fluid stress reduces to a pressure,

$$T_f^0 = -p_f I, \quad (3.34)$$

and from (2.11), the chemical potential tensor of the fluid becomes

$$K_f = \mu_f I - V \otimes \frac{\partial \psi_f}{\partial V}. \quad (3.35)$$

At a fluid-permeable surface considered in the previous section, from the relation (2.14), we have the following energy jump condition,

$$\left[ \left( K_f + \frac{1}{2} V_f^2 I \right) \rho_f V_f \right] \cdot \mathbf{n} = 0,$$

At this surface,  $\mathbf{v}_s = \mathbf{u}^*$ , so that  $V_f = \mathbf{v}_f - \mathbf{u}^* = V$ . Hence, by the use of (3.35), the condition becomes

$$\left[ \left[ \mu_f + \frac{1}{2} V^2 - V \cdot \frac{\partial \psi_f}{\partial V} \right] \rho_f V_f \right] \cdot \mathbf{n} = 0,$$

where we have noted that from the mass jump condition (2.12)<sub>1</sub> at the fluid-permeable surface ( $\rho_f V_f \cdot \mathbf{n}$ ) is continuous, and since it is not zero in general, it follows that

$$\left[ \left[ \mu_f + \frac{1}{2} V^2 - V \cdot \frac{\partial \psi_f}{\partial V} \right] \right] = 0. \quad (3.36)$$

In particular, we have

$$[\mu_f] \rightarrow 0 \quad \text{when} \quad V \rightarrow 0.$$

Thus, we conclude that in equilibrium ( $\mathbf{v}_f = \mathbf{u}^*$ ), the chemical potential of the fluid constituent is continuous across a fluid-permeable surface. This confirms the well-known result of the classical theory (see [4]).

## 4 Saturated porous media

The solid-fluid mixture considered in the previous section can be regarded as a model for saturated porous media provided that the concept of porosity is introduced. For mixture theory of porous media, a material point is regarded as a representative volume element  $dV$  which contains pores through them fluid constituent can flow. Physically, it is assumed that a representative volume element is large enough compare to solid grains (connected or not), yet at the same time small enough compare to the characteristic length of the material body.

Let the volume fraction of pores be denoted by  $\phi$ , then the volume fractions of the fluid and the solid are

$$dV_f = \phi dV, \quad dV_s = (1 - \phi) dV,$$

if the porous medium is saturated.

Remember that in the mixture theory, the mass densities are defined relative to the mixture volume, so that the fluid and solid mass in the representative volume element are given by

$$\begin{aligned} dM_f &= \rho_f dV = d_f dV_f, \\ dM_s &= \rho_s dV = d_s dV_s, \end{aligned}$$

and hence,

$$\rho_f = \phi d_f, \quad \rho_s = (1 - \phi) d_s, \quad (4.1)$$

where  $d_f$  and  $d_s$  are the true mass densities of fluid and solid constituents respectively.

We shall also regard the partial fluid pressure  $p_f$  in the mixture theory as the outcome of a “microscopic” pressure acting over the area fraction of surface actually occupied by the fluid in the pore, i.e.,

$$p_f dA = P dA_f, \quad \text{hence,} \quad p_f = \phi_a P,$$

where  $P$  will be called the *pore fluid pressure* and  $\phi_a = dA_f/dA$  is the area fraction of the pores.

In general, the volume fraction  $\phi$  and the area fraction  $\phi_a$  may be different, yet for practical applications, we shall adopt a reasonable assumption that they are the same for simplicity, so that the pore fluid pressure is defined as

$$P = \frac{p_f}{\phi}. \quad (4.2)$$

The pore pressure is an important concept in soil mechanics, we shall see that the definition (4.2) leads to results consistent with the well-known results in soil mechanics.

## 4.1 Pore fluid pressure

In most applications, it is reasonable to assume that in equilibrium the free energy of the fluid constituent is the same as the free energy of the pure fluid, i.e., we shall assume that

$$\psi_f^0 = \widehat{\psi}_f(\theta, d_f). \quad (4.3)$$

From (3.33), we have

$$\mu_f = \frac{P}{d_f} + \psi_f^0.$$

Since the true density does not change,  $[[d_f]] = 0$ , at an ideal fluid-permeable surface which allows the fluid to go through, it follows that

$$[[\mu_f]] = \left[ \left[ \frac{P}{d_f} \right] \right] + \left[ \left[ \widehat{\psi}_f(\theta, d_f) \right] \right] = \frac{1}{d_f} [P].$$

Therefore the condition (3.36) implies that

$$[P] + d_f \left[ \left[ \frac{1}{2}(\mathbf{v}_f - \mathbf{u}^*)^2 - (\mathbf{v}_f - \mathbf{u}^*) \cdot \frac{\partial \psi_f}{\partial \mathbf{V}} \right] \right] = 0.$$

In particular, if  $\mathbf{v}_f = \mathbf{u}^*$ , then

$$[P] = 0. \quad (4.4)$$

This result agrees with our physical intuition that if the fluid does not flow from one side to the other side through the pores then the pressures in the pores on both sides must be equal.

A typical fluid pressure measurement is the use of manometric tube attached to the specimen. In the case of a porous body, the junction of the manometric tube and the porous medium can be treated as a fluid-permeable ideal singular surface, where on one side is a solid-fluid mixture and on the other is a pure fluid at rest. Consequently by virtue of the jump condition (4.4) in equilibrium, the fluid pressure measured from the manometric tube is the pore fluid pressure (see [4]).

## 4.2 Equations of motion

The equations of motion (3.2)<sub>3,4</sub> for the fluid and the solid constituents in porous media can be written as

$$\begin{aligned}\phi d_f \dot{\mathbf{v}}_f &= \operatorname{div} T_f + \mathbf{m}_f + \phi d_f \mathbf{g}, \\ (1 - \phi) d_s \dot{\mathbf{v}}_s &= \operatorname{div} T_s - \mathbf{m}_f + (1 - \phi) d_s \mathbf{g},\end{aligned}$$

where  $\dot{\mathbf{v}}_f$  and  $\dot{\mathbf{v}}_s$  are the accelerations of the fluid and the solid respectively, and the external body force is the gravitational force  $\mathbf{g}$ .

Let us write the stresses in the following form,

$$\begin{aligned}T_f &= -\phi PI + \bar{T}_f, \\ T_s &= -(1 - \phi)PI + \bar{T}_s.\end{aligned}\tag{4.5}$$

We call  $\bar{T}_f$  the *extra fluid stress* and  $\bar{T}_s$  the *effective solid stress*, since it reduces to the effective stress widely used in soil mechanics as we shall see later. The equations of motion then become

$$\begin{aligned}\phi d_f \dot{\mathbf{v}}_f &= -\phi \operatorname{grad} P - P \operatorname{grad} \phi + \operatorname{div} \bar{T}_f + \mathbf{m}_f + \phi d_f \mathbf{g}, \\ (1 - \phi) d_s \dot{\mathbf{v}}_s &= -(1 - \phi) \operatorname{grad} P + P \operatorname{grad} \phi + \operatorname{div} \bar{T}_s - \mathbf{m}_f + (1 - \phi) d_s \mathbf{g}.\end{aligned}\tag{4.6}$$

On the other hand, note that the interactive force  $\mathbf{m}_f$  in equilibrium from (3.32), with the use of (3.33), can be written as

$$\mathbf{m}_f^0 = \frac{p_f}{\rho_f} \operatorname{grad} \rho_f - \rho_f (\operatorname{grad} \psi_f^0)^0,\tag{4.7}$$

which now becomes

$$\mathbf{m}_f^0 = P \operatorname{grad} \phi + \phi \left( \frac{P}{d_f} - d_f \frac{\partial \hat{\psi}_f}{\partial d_f} \right) \operatorname{grad} d_f,\tag{4.8}$$

by the use of (4.2) and (4.3).

## 4.3 Linear theory and Darcy's law

Since equilibrium is characterized by the conditions,  $\operatorname{grad} \theta = 0$  and  $V = 0$ , in a linear theory, we shall assume that  $|\operatorname{grad} \theta|$  and  $|V|$  are small quantities.

We shall call the non-equilibrium part of the interactive force

$$\mathbf{r} = \mathbf{m}_f - \mathbf{m}_f^0\tag{4.9}$$

the *resistive force*, since it is the force against the flow of the fluid through the medium. One may also call  $-\mathbf{r}$  the *drag force* acting upon the solid constituent. Furthermore, since it vanishes in equilibrium, we can write the resistive force as

$$\mathbf{r} = -\phi R V - \phi G \operatorname{grad} \theta + o(2),\tag{4.10}$$

where material parameters  $R$  and  $G$  are tensor-valued functions in general.

On the other hand, from (3.31) and (4.5), the extra fluid stress,

$$\bar{T}_f = \rho_f (\psi_f - \psi_f^0) I + \rho_f \frac{\partial \psi_f}{\partial V} \otimes V \approx o(2),$$

is a second order quantity because the free energy of fluid constituent must be a scalar-valued isotropic function.

Therefore, by the use of (4.8) through (4.10), the equations of motion for the fluid constituent (4.6)<sub>1</sub> in the linear theory becomes

$$d_f \dot{\mathbf{v}}_f = -\text{grad } P - \boldsymbol{\kappa} + d_f \mathbf{g}, \quad (4.11)$$

where

$$\boldsymbol{\kappa} = RV + G \text{grad } \theta - \left( \frac{P}{d_f} - d_f \frac{\partial \hat{\psi}_f}{\partial d_f} \right) \text{grad } d_f. \quad (4.12)$$

Similarly, the equation (4.6)<sub>2</sub> becomes

$$(1 - \phi) d_s \dot{\mathbf{v}}_s = -(1 - \phi) \text{grad } P + \text{div } \bar{T}_s + \phi \boldsymbol{\kappa} + (1 - \phi) d_s \mathbf{g}, \quad (4.13)$$

which is the equation of motion of the solid constituent.

We can obtain an interesting equation for the solid constituent if we multiply the equation (4.11) by  $(1 - \phi)$  and subtract it from the equation (4.13),

$$(1 - \phi)(d_s \dot{\mathbf{v}}_s - d_f \dot{\mathbf{v}}_f) = \text{div } \bar{T}_s + \boldsymbol{\kappa} + (1 - \phi)(d_s - d_f) \mathbf{g}. \quad (4.14)$$

In quasi-static case when the accelerations can be neglected, the two equations (4.11) and (4.14) for the fluid and the solid constituents can be written as

$$\begin{aligned} \text{grad } P + \boldsymbol{\kappa} &= d_f \mathbf{g}, \\ -\text{div } \bar{T}_s - \boldsymbol{\kappa} &= (1 - \phi)(d_s - d_f) \mathbf{g}. \end{aligned} \quad (4.15)$$

Note that the term  $(1 - \phi)(d_s - d_f) \mathbf{g}$ , which represents the difference between the gravitational force of the solid and the fluid in the volume fraction of the solid, can be regarded as the *buoyancy force* on the solid constituent.

Furthermore, adding the two equations in (4.15) together, we obtain

$$-\text{div } T = \rho \mathbf{g}. \quad (4.16)$$

Clearly this is the equilibrium equation for the mixture as a whole, where from (4.5)

$$T = T_f + T_s = -PI + \bar{T}_s + o(2), \quad \rho = \phi d_f + (1 - \phi) d_s, \quad (4.17)$$

are the total stress and the total mass density of the mixture respectively.

**Remark.** The equation of motion for the fluid constituent (4.11) is a generalization of *Darcy's law*. Indeed, it is shown in [4], that for the case of classical Darcy's experiment (see [11]), the equation reduces to the original form of the Darcy's law.

In particular, if only the essential contribution in (4.12)  $\boldsymbol{\kappa} = RV$ , is taken into account and the solid matrix is stationary  $\mathbf{v}_s = 0$ , the equation (4.15)<sub>1</sub> become

$$\mathbf{v}_f = -R^{-1}(\text{grad } P - d_f \mathbf{g}). \quad (4.18)$$

Roughly speaking, it asserts that pore pressure gradient is the driving force for the flow of the fluid through the medium. This is the Darcy's law in the usual classical form, where the reciprocal of the resistivity constant  $R$  is call the *permeability tensor*.

## 5 Incompressible porous media

We shall consider incompressible porous media, i.e., for which the true mass densities  $d_f$  and  $d_s$  are constants. For a linear theory, we have

$$\begin{aligned} T_f &= -\phi PI, \\ T_s &= -(1 - \phi)PI + \bar{T}_s, \\ \boldsymbol{\kappa} &= RV + G \text{grad } \theta. \end{aligned} \quad (5.1)$$

where the pore pressure  $P$  is constitutively indeterminate and the effective solid stress is given by the constitutive relation,  $\bar{T}_s = \bar{T}_s(\phi, F_s)$ .



## 5.1 Equilibrium pore pressure

From (4.15), the equations in equilibrium ( $\kappa = 0$ ) become

$$\begin{aligned} \text{grad } P &= d_f \mathbf{g}, \\ -\text{div } \bar{T}_s &= (1 - \phi)(d_s - d_f) \mathbf{g}. \end{aligned} \quad (5.2)$$

These two equations can be solved separately. Suppose that the  $x$ -coordinate is in the vertical downward direction,  $\mathbf{g} = g \mathbf{e}_x$ . Then the first equation can be integrated immediately to give

$$P = d_f g x + P_0, \quad P_0 = P(0). \quad (5.3)$$

This result asserts that the equilibrium pore pressure is the hydrostatic pressure. It agrees with the observation in soil mechanics from experimental measurements that the manometric pressure in the soil is the pressure as if the medium were bulk fluid, unaffected by the presence of the solid constituent in the medium.

We remark that this result is sometimes overlooked in the mixture theory of porous media. It is mainly due to the fact that in the theory of *simple mixture* (in the sense of Noll's simple materials), which omits the second gradients of deformations as independent constitutive variables, the equilibrium interactive force  $\mathbf{m}^0$  is identically zero by constitutive hypothesis (see (3.32)). However, from the relation (4.8),  $\mathbf{m}_0$  is not a negligible quantity for a body with non-uniform porosity,  $\mathbf{m}_0 = P \text{grad } \phi$ , and it is easy to see that in the absence of this term, the result (5.3) need not follow. This remark, we shall regard as a strong evidence that porous media must be treated as *non-simple mixtures* even for a linear equilibrium theory.

## 5.2 Effective stress principle

From (5.1), we have

$$T = T_f + T_s = -PI + \bar{T}_s,$$

where the pore pressure  $P$  is given by (5.3).

Therefore, the effective stress of the solid constituent is the part of the total stress in excess of the pore pressure (see also (4.17)<sub>1</sub> in more general case). This is known as the *effective stress principle* in soil mechanics first introduced by Terzaghi (see [12]).

Both the concepts of pore pressure and effective stress are elementary in soil mechanics. However, regarding soil as porous media in terms of the theories of mixtures, such results are often overlooked without giving due attention.

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