Remarks on Material Frame-Indifference Controversy

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Abstract

Regarding a frame of reference as an observer, the notion of frame-indifference (also referred to as objectivity) concerns the transformation properties of physical quantities under change of observer. For kinematic quantities, they can usually be derived from the deformation/motion, while for non-kinematic quantities, such as force and stress, frame-indifference property must be postulated. Frame-indifference postulate for the stress, sometimes unsuitably called the principle of frame-indifference, is a universal assumption which has nothing to do with material properties. This has caused some confusions in the interpretation of "material" frame-indifference in the literature. Material frame-indifference deals with the constitutive functions which characterize intrinsic properties of the material under different observers. We shall carefully render these concepts mathematically and deduce the well-known condition of material objectivity as a consequence of the frame-indifference postulate and the principle of material frame-indifference. We shall also emphasize and remark on some persistent controversy and some misleading statements found in recent literature.

Mathematics Subject Classification: 74A05, 74A20, 74A99, 76A99.

Keywords: Observer, frame of reference, change of frame, material frame-indifference, constitutive functions, isometry of Euclidean spaces, Euclidean transformation.

1 Introduction

The foundation of Modern Continuum Mechanics is based on the fundamental ideas, set forth in the biblical treatise, *The Non-Linear Field Theories of Mechanics*, by Truesdell and Noll [17], among which, one of the most important and controversial ones is the principle of material frame-indifference (MFI). Due to the original somewhat loose statements, attempts for better interpretation of MFI appeared again and again in the literature and some controversy persisted to these days [1, 2, 4, 11, 13, 16, 18].

As we understand, the essential meaning of MFI is the simple idea that material properties are independent of observers. In order to explain this, one has to know what a frame of reference (regarded as an observer) is, so as to define what frame-indifference means. However, frame-indifference concerns only transformation properties of physical quantities under change of frame, and hence has nothing to do with the material properties. On the other hand, material frame-indifference, which deals with constitutive functions characterizing intrinsic properties of the material under different observers, is postulated as a guiding principle for the formulation of constitutive models. Therefore, frame-indifference and material frame-indifference are two different concepts not to be confused. We shall make these different concepts mathematically clear in this paper.

A similar version of this paper was included in a conference proceedings [8]. Nevertheless, the present one has been more carefully refined and it is one of our objective to make additional remarks and comments on some persistent controversy and misleading statements found in recent literature.

Conventional notations now widely used in continuum mechanics textbooks will be followed. For consideration of allowing different observers possibly belong to different Euclidean spaces as sometimes suggested ([11]), our formulations take this into consideration, for which we introduce some notions of isometries in Euclidean spaces with changes in orientation as well as scaling in the Appendix, in which examples are given to justify the rational for such consideration. Please also refer to the Appendix for relevant definitions and notations concerning isometric transformations.

2 Frame of reference

The event world $\mathcal{W}$ is a four-dimensional space-time in which physical events occur at some places and certain instants. Let $\mathcal{T}$ be the collection of instants and $\mathcal{W}_s$ be the placement space of simultaneous events at the instant s, then the neo-classical space-time (Noll [12]) can be expressed as the disjoint union of placement spaces of simultaneous events at each instant,

$$\mathcal{W} = \bigcup_{s \in \mathcal{T}} \mathcal{W}_s \,.$$

A point $p_s \in \mathcal{W}$ is called an event, which occurs at the instant s and the place $p \in \mathcal{W}_s$. At different instants s and $\bar{s}$, the spaces $\mathcal{W}_s$ and $\mathcal{W}_{\bar{s}}$ are two disjoint spaces. Thus it is impossible to determine the distance between two non-simultaneous events at p_s and $p_{\bar{s}}$ if $s \neq \bar{s}$, and hence $\mathcal{W}$ is not a product space of space and time. However, it can be set into correspondence with a product space through a frame of reference on $\mathcal{W}$.

Definition. *(Frame of reference): A frame of reference is a one-to-one mapping*

$$\phi : \mathcal{W} \to \mathbb{E} \times \mathbb{R},$$

taking $p_s \mapsto (\boldsymbol{x}, t)$*, where* $\mathbb{R}$ *is the space of real numbers and* $\mathbb{E}$ *is a three-dimensional Euclidean space. We shall denote the map taking* $p \mapsto \boldsymbol{x}$ *as the map* $\phi_s : \mathcal{W}_s \to \mathbb{E}$.

In general, the Euclidean spaces of different frames of reference may not be the same. Therefore, for definiteness, we shall denote the Euclidean space of the frame ϕ by $\mathbb{E}_\phi$, and its translation space by $\mathbb{V}_\phi$. We assume that $\mathbb{V}_\phi$ is an inner product space.

Of course, there are infinite many frames of reference. Each one of them may be regarded as an *observer*, since it can be depicted as a person taking a snapshot so that the image of ϕ_s is a picture (three-dimensional at least conceptually) of the placements of the events at some instant s, from which the distance between two simultaneous events can be measured. A sequence of events can also be recorded as video clips depicting the change of events in time by an observer.

Now, suppose that two observers are recording the same events with video cameras. In order to compare their video clips regarding the locations and time, they must have a mutual agreement that the clock of their cameras must be synchronized so that simultaneous events can be recognized and since during the recording two observers may move independently while taking pictures with their cameras from different angles, there will be a relative motion, a scaling and a relative orientation between them. We shall make such a consensus among observers explicit mathematically.

Let ϕ and ϕ^* be two frames of reference. They are related by the composite map $* := \phi^* \circ \phi^{-1}$,

$$* : \mathbb{E}_\phi \times \mathbb{R} \to \mathbb{E}_{\phi^*} \times \mathbb{R}, \quad \text{taking} \quad (\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*),$$

where $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ are the position and time of the same event observed by ϕ and ϕ^* simultaneously. In general $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$ are different Euclidean spaces. Physically, not any change of frame

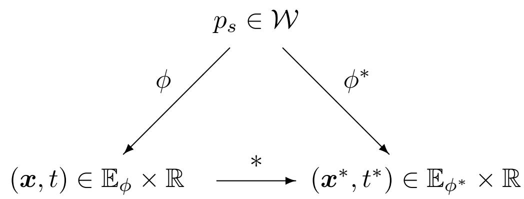

Figure 1: A change of frame

would be of our concern as long as we are interested in establishing a consensus among observers, which should require preservation of distance between simultaneous events and time interval as well as the sense of time.

Definition. *(Euclidean change of frame): A change of frame (observer) from ϕ to ϕ^* taking $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*)$, is an isometry of space and time given by*

$$\boldsymbol{x}^* = \mathcal{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}^*(t), \qquad t^* = t + a, \tag{1}$$

for some constant time difference $a \in \mathbb{R}$, some relative translation $\boldsymbol{c}^* : \mathbb{R} \to \mathbb{E}_{\phi^*}$ with respect to the reference point $\boldsymbol{x}_0 \in \mathbb{E}_\phi$ and some isometric transformation $\mathcal{Q} : \mathbb{R} \to \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$.

We have denoted $\mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*}) = \{\mathcal{Q} \in \mathcal{L}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*}) : \|\mathcal{Q}\boldsymbol{u}\|_{\mathbb{V}_{\phi^*}} = \|\boldsymbol{u}\|_{\mathbb{V}_\phi}, \ \forall \boldsymbol{u} \in \mathbb{V}_\phi\}$. Euclidean changes of frame will often be called *changes of frame* for simplicity, since they are the only changes of frame among consenting observers of our concern for the purpose of discussing frame-indifference in continuum mechanics.

All consenting observers form an equivalent class, denoted by $\mathfrak{E}$, among the set of all observers, i.e., for any $\phi, \phi^* \in \mathfrak{E}$, there exists a Euclidean change of frame from $\phi \to \phi^*$. From now on, only classes of consenting observers will be considered. Therefore, any observer, would mean any observer in some $\mathfrak{E}$, and a change of frame, would mean a Euclidean change of frame.

3 Motion and deformation

In the space-time, a physical event is represented by its placement at a certain instant so that it can be observed in a frame of reference. Let a body $\mathcal{B}$ be a set of material points.

Definition. *(Configuration): Let $\xi : \mathcal{B} \to \mathcal{W}_t$ be a placement of the body $\mathcal{B}$ at the instant t, and let ϕ be a frame of reference, then the composite map $\xi_{\phi_t} := \phi_t \circ \xi$,*

$$\xi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi$$

is called a configuration of the body $\mathcal{B}$ at the instant t in the frame ϕ.

A configuration thus identifies the body with a region in the Euclidean space of the observer. In this sense, the motion of a body can be viewed as a continuous sequence of events such that at any instant t, the placement of the body $\mathcal{B}$ in $\mathcal{W}_t$ is a one-to-one mapping

$$\chi_t : \mathcal{B} \to \mathcal{W}_t,$$

and the composite mapping $\chi_{\phi_t} := \phi_t \circ \chi_t$,

$$\chi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi_{\phi_t}(p) = \phi_t(\chi_t(p)), \quad p \in \mathcal{B},$$

is the configuration of the body in the motion χ_t. Let $\mathcal{B}_{\chi_t} := \chi_{\phi_t}(\mathcal{B}) \subset \mathbb{E}_\phi$ (see the right part of Figure 2). The motion can then be regarded as a sequence of configurations of $\mathcal{B}$ in time, $\chi_\phi = \{\chi_{\phi_t}, t \in \mathbb{R} \mid \chi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi\}$. We can also express a motion as

$$\chi_\phi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi_\phi(p, t) = \chi_{\phi_t}(p), \quad p \in \mathcal{B}.$$

Note that in our discussions, we have been using $t \in \mathbb{R}$ as the time in the frame ϕ corresponding to the instant $s \in \mathcal{T}$ with $s = t$ for simplicity without loss of generality.

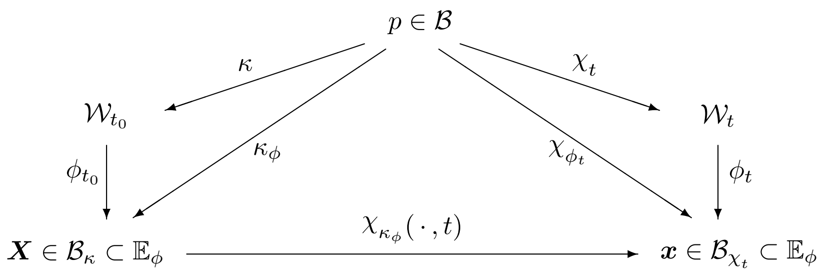

Figure 2: Motion χ_{ϕ_t}, reference configuration κ_ϕ and deformation $\chi_{\kappa_\phi}(\,\cdot\,,t)$

Reference configuration

We regard a body $\mathcal{B}$ as a set of material points. Although it is possible to endow the body as a manifold with a differentiable structure and topology for doing mathematics on the body, to avoid such mathematical subtleties, usually a particular configuration is chosen as reference (see the left part of Figure 2),

$$\kappa_\phi : \mathcal{B} \to \mathbb{E}_\phi, \qquad \boldsymbol{X} = \kappa_\phi(p), \qquad \mathcal{B}_\kappa := \kappa_\phi(\mathcal{B}) \subset \mathbb{E}_\phi,$$

so that the motion at an instant t is a one-to-one mapping

$$\chi_{\kappa_\phi}(\cdot, t) : \mathcal{B}_\kappa \to \mathcal{B}_{\chi_t}, \qquad \boldsymbol{x} = \chi_{\kappa_\phi}(\boldsymbol{X}, t) = \chi_{\phi_t}(\kappa_\phi^{-1}(\boldsymbol{X})), \quad \boldsymbol{X} \in \mathcal{B}_\kappa,$$

from a region into another region in the same Euclidean space $\mathbb{E}_\phi$ for which topology and differentiability are well defined. This mapping is called a *deformation* from κ to χ_t in the frame ϕ and a motion is then a sequence of deformations in time.

For the reference configuration κ_ϕ, there is some instant, say t_0, at which the reference placement of the body is chosen, $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ (see Figure 2). On the other hand, the choice of a reference configuration is arbitrary, and it is not necessary that the body should actually occupy the reference place in its motion under consideration. Nevertheless, in most practical problems, t_0 is usually taken as the initial time of the motion.

4 Frame-indifference

The change of frame (1) gives rise to a linear mapping on the translation space, in the following way: Let $\boldsymbol{u}(\phi) = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in \mathbb{V}_\phi$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}_\phi$ in the frame ϕ, and $\boldsymbol{u}(\phi^*) = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in \mathbb{V}_{\phi^*}$ be the corresponding difference vector in the frame ϕ^*, then from (1), it follows immediately that

$$\boldsymbol{u}(\phi^*) = \mathcal{Q}(t)\boldsymbol{u}(\phi),$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation associated with the change of frame $\phi \to \phi^*$.

Any vector quantity in $\mathbb{V}_\phi$, which has this transformation property, is said to be objective with respect to Euclidean transformations. This concept of objectivity can be generalized to any tensor spaces of $\mathbb{V}_\phi$.

Let

$$s : \mathfrak{E} \to \mathbb{R}, \qquad \boldsymbol{u} : \mathfrak{E} \to \mathbb{V}_\mathfrak{E}, \qquad T : \mathfrak{E} \to \mathbb{V}_\mathfrak{E} \otimes \mathbb{V}_\mathfrak{E},$$

where $\mathfrak{E}$ is the Euclidean class of frames of reference and $\mathbb{V}_{\mathfrak{E}} = \{\mathbb{V}_\phi : \phi \in \mathfrak{E}\}$. They are scalar, vector and (second order) tensor *observable* quantities respectively. We call $f(\phi)$ the value of the quantity f observed in the frame ϕ.

Definition. *(Frame-indifference): Relative to a change of frame from ϕ to ϕ^*, the observables s, $\boldsymbol{u}$ and T are called frame-indifferent (or objective) scalar, vector and tensor quantities respectively, if they satisfy the following transformation properties:*

$$s(\phi^*) = s(\phi),$$
$$\boldsymbol{u}(\phi^*) = \mathcal{Q}(t)\,\boldsymbol{u}(\phi),$$
$$T(\phi^*) = \mathcal{Q}(t)\,T(\phi)\,\mathcal{Q}(t)^\top,$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame from ϕ to ϕ^*.

More precisely, they are also said to be frame-indifferent with respect to Euclidean transformations or simply Euclidean objective. For simplicity, we often write $f = f(\phi)$ and $f^* = f(\phi^*)$.

Transformation properties of motion

Let χ_ϕ be a motion of the body in the frame ϕ, and χ_{ϕ^*} be the corresponding motion in ϕ^*,

$$\boldsymbol{x} = \chi_\phi(p,t), \quad \boldsymbol{x}^* = \chi_{\phi^*}(p,t^*), \quad p \in \mathcal{B}.$$

Then from (1), we have

$$\chi_{\phi^*}(p,t^*) = \mathcal{Q}(t)\big(\chi_\phi(p,t) - \boldsymbol{x}_o\big) + \boldsymbol{c}^*(t), \quad p \in \mathcal{B},$$

from which, one can easily show that the velocity and the acceleration are not objective quantities,

$$\dot{\boldsymbol{x}}^* = \mathcal{Q}\dot{\boldsymbol{x}} + \dot{\mathcal{Q}}(\boldsymbol{x} - \boldsymbol{x}_o) + \dot{\boldsymbol{c}}^*,$$
$$\ddot{\boldsymbol{x}}^* = \mathcal{Q}\ddot{\boldsymbol{x}} + 2\dot{\mathcal{Q}}\dot{\boldsymbol{x}} + \ddot{\mathcal{Q}}(\boldsymbol{x} - \boldsymbol{x}_0) + \ddot{\boldsymbol{c}}^*. \tag{2}$$

A change of frame (1) with constant $\mathcal{Q}$ and $\boldsymbol{c}^*(t) = \boldsymbol{c}_0 + \boldsymbol{c}_1 t$, for constant $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ (so that $\dot{\mathcal{Q}} = 0$ and $\ddot{\boldsymbol{c}}^* = 0$), is called a *Galilean transformation*. Therefore, from (2) we conclude that the acceleration is not Euclidean objective but it is objective with respect to Galilean transformation. Moreover, it also shows that the velocity is neither a Euclidean nor a Galilean objective vector quantity.

Transformation properties of deformation gradient

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Figure 3), then

$$\kappa_\phi = \phi_{t_0} \circ \kappa \quad \text{and} \quad \kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa \tag{3}$$

are the corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant, and

$$\boldsymbol{X} = \kappa_\phi(p), \quad \boldsymbol{X}^* = \kappa_{\phi^*}(p), \quad p \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows from (3) that $\gamma = \phi^*_{t_0} \circ \phi_{t_0}^{-1}$ and by (1), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = \mathcal{Q}(t_0)(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}^*(t_0). \tag{4}$$

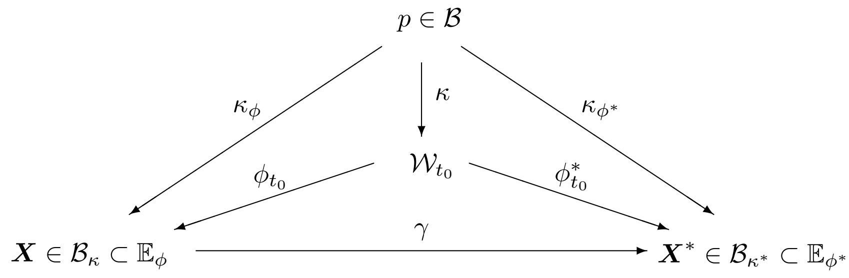

Figure 3: Reference configurations κ_ϕ and κ_{ϕ^*} in the change of frame from ϕ to ϕ^*

On the other hand, the motion in referential description relative to the change of frame is given by $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ and $\boldsymbol{x}^* = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*)$. Hence from (1), we have

$$\chi_{\kappa^*}(\boldsymbol{X}^*, t^*) = \mathcal{Q}(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_o) + \boldsymbol{c}^*(t).$$

Therefore we obtain for the deformation gradients, $F = \nabla_{\boldsymbol{X}}\chi_\kappa$ in the frame ϕ and $F^* = \nabla_{\boldsymbol{X}^*}\chi_{\kappa^*}$ in the frame ϕ^*, by the chain rule and the use of (4),

$$F^*(\boldsymbol{X}^*, t^*) = \mathcal{Q}(t)F(\boldsymbol{X}, t)\,\mathcal{Q}(t_0)^\top, \quad \text{or simply} \quad F^* = \mathcal{Q}F\mathcal{Q}_0^\top, \tag{5}$$

where $\mathcal{Q}_0 = \mathcal{Q}(t_0)$ is the isometric transformation due to the change of frame at the instant t_0 when the reference configuration is chosen.

Remark 1. The transformation property (5) stands in contrast to $F^* = \mathcal{Q}F$, the widely used formula which is obtained "provided that the reference configuration is unaffected by the change of frame" as usually implicitly assumed (see [17] p. 308), so that $\mathcal{Q}_0$ reduces to the identity transformation. On the other hand, the transformation property $F^* = \mathcal{Q}F\mathcal{Q}_0^\top$ has also been derived elsewhere in the literature ([10],[15] Sec. 2.2.8). □

Remark 2. Keep in mind that a configuration is a placement of a body *relative* to a frame of reference and any *reference* configuration is not an exception. Therefore, to say that the "reference configuration is unaffected by the change of frame" is at best an assumption ([2] Sec. 20.1). Oftentimes, in order to justify this assumption, it is remarked that since the transformation $\mathcal{Q}(t)$ in the change of frame can be chosen arbitrarily, one may presume that at some instant t_0 it is an identity transformation. This argument may seem quite enticing, however, it cannot be valid when $\mathcal{Q}$ is simply a constant transformation as in the case of Galilean change of frame.

On the other hand, two consenting observers cannot independently choose their reference configurations, because they must choose the configuration of the body at the *same* instant of their respective frames of reference (see discussions in [5, 11]). □

5 Galilean invariance of balance laws

In classical mechanics, Newton's first law, often known as the law of inertia, is essentially a definition of inertial frame.

Definition. *(Inertial frame): A frame of reference is called an inertial frame if, relative to it, the velocity of a body remains constant unless the body is acted upon by an external force.*

We present the first law in this manner in order to emphasize that the existence of inertial frames is essential for the formulation of Newton's second law, which asserts that relative to an inertial frame, the equation of motion takes the simple form:

$$\mathfrak{m}\,\ddot{\boldsymbol{x}} = \mathfrak{f}. \tag{6}$$

Now, we shall assume that there is an inertial frame $\phi_0 \in \mathfrak{E}$, for which the equation of motion of a particle is given by (6), and we are interested in how the equation is transformed under a change of frame.

Unlike the acceleration, transformation properties of non-kinematic quantities cannot be deduced theoretically. Instead, for the mass $\mathfrak{m}$ and the force $\mathfrak{f}$, it is conventionally *postulated* that they are Euclidean objective scalar and vector quantities respectively, so that for any change from ϕ_0 to $\phi^* \in \mathfrak{E}$ given by (1), we have

$$\mathfrak{m}^* = \mathfrak{m}, \qquad \mathfrak{f}^* = \mathcal{Q}\,\mathfrak{f},$$

which together with (2), by multiplying (6) with $\mathcal{Q}$, we obtain the equation of motion in the (non-inertial) frame ϕ^*,

$$\mathfrak{m}^*\ddot{\boldsymbol{x}}^* = \mathfrak{f}^* + \mathfrak{m}^*\,\mathfrak{i}^*, \tag{7}$$

where $\mathfrak{i}^*$ is called the inertial force given by

$$\mathfrak{i}^* = \ddot{\boldsymbol{c}}^* + 2\Omega(\dot{\boldsymbol{x}}^* - \dot{\boldsymbol{c}}^*) + (\dot{\Omega} - \Omega^2)(\boldsymbol{x}^* - \boldsymbol{c}^*),$$

where $\Omega = \dot{\mathcal{Q}}\mathcal{Q}^\top : \mathbb{R} \to L(\mathbb{V}_{\phi^*})$ is called the spin tensor of the frame ϕ^* relative to the inertial frame ϕ_0.

Note that the inertial force vanishes if the change of frame $\phi_0 \to \phi^*$ is a Galilean transformation, i.e., $\dot{\mathcal{Q}} = 0$ and $\ddot{\boldsymbol{c}}^* = 0$, and hence the equation of motion in the frame ϕ^* also takes the simple form,

$$m^*\ddot{\boldsymbol{x}}^* = \mathfrak{f}^*,$$

which implies that the frame ϕ^* is also an inertial frame.

Therefore, any frame of reference obtained from a Galilean change of frame from an inertial frame is also an inertial frame. Hence, all inertial frames form an equivalent class $\mathfrak{G}$, such that for any $\phi, \phi^* \in \mathfrak{G}$, the change of frame $\phi \to \phi^*$ is a Galilean transformation. The Galilean class $\mathfrak{G}$ is a subclass of the Euclidean class $\mathfrak{E}$.

Remark 3. Since Euclidean change of frame is an equivalence relation, it decomposes all frames of reference into disjoint equivalence classes, i.e., Euclidean classes as we previously called. However, the existence of an inertial frame which is essential in establishing dynamic laws in mechanics, leads to a special choice of Euclidean class of interest.

Let $\mathfrak{E}$ be the Euclidean class which contains an inertial frame. Since different Euclidean classes are not related by any Euclidean transformation, hence, nor by any Galilean transformation, it is obvious that the Euclidean class $\mathfrak{E}$ is the only class containing the subclass $\mathfrak{G}$ of all inertial frames. Consequently, from now on, the only Euclidean class of interest for further discussions, is the one, denoted by $\mathfrak{E}$, containing Galilean class of all inertial frames. □

In short, we can assert that physical laws, like the equation of motion, are in general not (Euclidean) frame-indifferent. Nevertheless, the equation of motion is Galilean frame-indifferent, under the assumption that mass and force are frame-indifferent quantities. This is usually referred to as *Galilean invariance* of the equation of motion.

Motivated by classical mechanics, the balance laws of mass, linear momentum, and energy for deformable bodies,

$$\begin{aligned}
&\dot{\rho} + \rho\,\operatorname{div}\dot{\boldsymbol{x}} = 0,\\
&\rho\,\ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\,\boldsymbol{b},\\
&\rho\,\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} - T\cdot\operatorname{grad}\dot{\boldsymbol{x}} = \rho\,r,
\end{aligned} \tag{8}$$

in an inertial frame are required to be invariant under Galilean transformation. Since two inertial frames are related by a Galilean transformation, it means that the equations (8) should hold in the same form in any inertial frame. In particular, the balance of linear momentum takes the forms in the inertial frames $\phi, \phi^* \in \mathfrak{G}$,

$$\rho\, \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\, \boldsymbol{b}, \qquad \rho^* \ddot{\boldsymbol{x}}^* - (\operatorname{div} T)^* = \rho^* \boldsymbol{b}^*.$$

Since the acceleration $\ddot{\boldsymbol{x}}$ is Galilean objective, in order this to hold, it is usually assumed that the mass density ρ, the Cauchy stress tensor T and the body force $\boldsymbol{b}$ are objective scalar, tensor, and vector quantities respectively. Similarly, for the energy equation, it is also assumed that the internal energy ε and the energy supply r are objective scalars, and the heat flux $\boldsymbol{q}$ is an objective vector. These assumptions concern the non-kinematic quantities, including external supplies $(\boldsymbol{b}, r)$, and the constitutive quantities $(T, \boldsymbol{q}, \varepsilon)$.

In fact, for Galilean invariance of the balance laws, only frame-indifference with respect to Galilean transformation for all those non-kinematic quantities would be sufficient. However, similar to classical mechanics, it is *postulated* that they are not only Galilean objective but also Euclidean objective. Therefore, with the known transformation properties of the kinematic variables, the balance laws in any arbitrary frame can be deduced.

To emphasize the importance of the objectivity postulate for constitutive theories, it will be referred to as Euclidean objectivity for constitutive quantities:

Euclidean objectivity. *The constitutive quantities: the Cauchy stress T, the heat flux $\boldsymbol{q}$ and the internal energy density ε, are Euclidean objective (Euclidean frame-indifferent),*

$$T(\phi^*) = \mathcal{Q}(t)\, T(\phi)\, \mathcal{Q}(t)^{\top}, \qquad \boldsymbol{q}(\phi^*) = \mathcal{Q}(t)\, \boldsymbol{q}(\phi), \qquad \varepsilon(\phi^*) = \varepsilon(\phi), \tag{9}$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame from ϕ to ϕ^*.

Note that this postulate concerns only frame-indifference properties of balance laws, so that it is a universal property for any deformable bodies, and therefore, do not concern any aspects of material properties of the body.

6 Constitutive equations in material description

Physically a state of thermomechanical behavior of a body is characterized by a description of the fields of density $\rho(p,t)$, motion $\chi(p,t)$ and temperature $\theta(p,t)$. The material properties of a body generally depend on the past history of its thermomechanical behavior.

Let us introduce the notion of the past history of a function. Let $h(\cdot)$ be a function of time. The history of h up to time t is defined by

$$h^t(s) = h(t-s),$$

where $s \in [\,0, \infty)$ denotes the time-coordinate pointing into the past from the present time t. Clearly $s = 0$ corresponds to the present time, therefore $h^t(0) = h(t)$.

Mathematical descriptions of material properties are called constitutive equations. We postulate that the history of thermomechanical behavior up to the present time determines the properties of the material body.

Principle of determinism. *Let ϕ be a frame of reference, and $\mathcal{C}$ be a constitutive quantity, then the constitutive equation for $\mathcal{C}$ is given by a functional of the form,*

$$\mathcal{C}(\phi, p, t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t; p), \qquad p \in \mathcal{B},\ t \in \mathbb{R}, \tag{10}$$

where the first three arguments are history functions:

$$\rho^t : \mathcal{B} \times [\,0, \infty) \to \mathbb{R}, \qquad \chi^t : \mathcal{B} \times [\,0, \infty) \to \mathbb{E}_\phi, \qquad \theta^t : \mathcal{B} \times [\,0, \infty) \to \mathbb{R}.$$

We call $\mathcal{F}_\phi$ the constitutive function of $\mathcal{C}$ in the frame ϕ. Such a functional allows the description of non-local effect of an inhomogeneous body with memory of past thermomechanical history. With the notation $\mathcal{F}_\phi$, we emphasize that the value of a constitutive function depends on the frame of reference ϕ in general.

For simplicity in further discussions on constitutive equations, we shall restrict our attention to material models for mechanical theory only, and only constitutive equation for the stress tensor will be considered. General results can be found elsewhere ([3, 6]).

In order to avoid possible confusions arisen from the viewpoint of employing different Euclidean spaces, we shall be more careful about expressing relevant physical quantities in the proper space.

Let the set of history functions on a set $\mathcal{X}$ in some space $\mathbb{W}$ be denoted by

$$\mathfrak{H}(\mathcal{X},\mathbb{W}) = \{h^t : \mathcal{X}\times[0,\infty)\to\mathbb{W}\}.$$

Then the constitutive equation for the stress tensor, $T(\phi,p,t)\in\mathbb{V}_\phi\otimes\mathbb{V}_\phi$, can be written as

$$T(\phi,p,t) = \mathcal{F}_\phi(\chi^t;p), \qquad \phi\in\mathfrak{E}, \quad p\in\mathcal{B}, \quad \chi^t\in\mathfrak{H}(\mathcal{B},\mathbb{E}_\phi). \tag{11}$$

Condition of Euclidean objectivity

Let $\phi^*\in\mathfrak{E}$ be another frame of reference, then the constitutive equation for the stress, $T(\phi^*,p,t^*)\in\mathbb{V}_{\phi^*}\otimes\mathbb{V}_{\phi^*}$, can be written as

$$T(\phi^*,p,t^*) = \mathcal{F}_{\phi^*}(*(\chi^t);p), \qquad p\in\mathcal{B}, \quad *(\chi^t)\in\mathfrak{H}(\mathcal{B},\mathbb{E}_{\phi^*}),$$

where the corresponding histories of motion are related by

$$*(\chi^t)(\bar p,s) = \mathcal{Q}^t(s)\big(\chi^t(\bar p,s)-\boldsymbol{x}_o\big)+\boldsymbol{c}^{*t}(s),$$

for any $s\in[0,\infty)$ and any $\bar p\in\mathcal{B}$, in the change of frame $\phi\to\phi^*$ given by (1).

We need to bear in mind that according to the assumption referred to as the Euclidean objectivity (9), the stress is a frame-indifferent quantity under a change of observer,

$$T(\phi^*,p,t^*) = \mathcal{Q}(t)T(\phi,p,t))\mathcal{Q}(t)^\top.$$

Therefore, it follows immediately that

$$\mathcal{F}_{\phi^*}(*(\chi^t);p) = \mathcal{Q}(t)\mathcal{F}_\phi(\chi^t;p)\mathcal{Q}(t)^\top, \qquad \chi^t\in\mathfrak{H}(\mathcal{B},\mathbb{E}_\phi), \tag{12}$$

where $\mathcal{Q}(t)\in\mathcal{O}(\mathbb{V}_\phi,\mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame $*:\phi\to\phi^*$.

The relation (12) will be referred to as the *condition of Euclidean objectivity*. It is a relation between the constitutive functions relative to two different observers. In other words, different observers cannot independently propose their own constitutive equations. Instead, the condition of Euclidean objectivity (12) determines the constitutive function $\mathcal{F}_{\phi^*}$ once the constitutive function $\mathcal{F}_\phi$ is given or vice-versa. They determine one from the other in a frame-dependent manner.

7 Principle of material frame-indifference

It is obvious that *not* any proposed constitutive equations would physically make sense as material models. First of all, they may be frame-dependent. However, since the constitutive functions must characterize the intrinsic properties of the material body itself, it should be observer-independence in certain sense. Consequently, there must be some restrictions imposed on the constitutive functions so that they would be indifferent to the change of frame. This is the essential idea of the principle of material frame-indifference.

Remark 4. In the case that does not distinguish the Euclidean spaces relative to different observers, i.e., $\mathbb{E}_\phi = \mathbb{E}_{\phi^*} = \mathbb{E}$ and $\mathbb{V}_\phi = \mathbb{V}_{\phi^*} = \mathbb{V}$, as adopted usually in the literature ([17, 3, 6]), the principle of material frame-indifference can simply be postulated as

$$\mathcal{F}_\phi(\,\bullet\,;p) = \mathcal{F}_{\phi^*}(\,\bullet\,;p), \qquad p \in \mathcal{B}. \tag{13}$$

where $\bullet$ represents any history of motion in $\mathfrak{H}(\mathcal{B},\mathbb{E})$ which is the common domain of the two functionals, and their values are in the same tensor space $\mathbb{V}\otimes\mathbb{V}$.

This states that for different observers $\phi,\phi^* \in \mathfrak{E}$, they all have the same constitutive function, $\mathcal{F}_\phi = \mathcal{F}_{\phi^*}$. Note that in (13) the material point p is superfluously indicated to emphasize that it is valid only when the *material description* is used, because in referential description, the choice of reference configuration may change material properties. For example, it is obvious that after a shear deformation, an isotropic material is no longer isotropic. Therefore, if such a deformed state is chosen as a reference configuration, the material property would have changed accordingly. □

Recall that a change of frame $* : \phi \to \phi^*$ is associated with an isometry between $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$ and conversely, given an isometry

$$i : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}, \qquad i(\boldsymbol{x}) = \mathcal{I}(t)(\boldsymbol{x} - \boldsymbol{x}_i) + \boldsymbol{x}_i^*(t), \tag{14}$$

there is a change of frame $\phi \to \phi_i$ taking $(\boldsymbol{x},t) \mapsto (i(\boldsymbol{x}),t)$. For this change of frame $\mathcal{I} \in \mathcal{O}(\mathbb{V}_\phi,\mathbb{V}_{\phi^*})$ and $\mathbb{E}_{\phi_i} = \mathbb{E}_{\phi^*}$. Therefore, the following condition of Euclidean objectivity (12) must hold,

$$\mathcal{F}_{\phi_i}(i(\chi^t);p) = \mathcal{I}(t)\,\mathcal{F}_\phi(\chi^t;p)\,\mathcal{I}(t)^\top, \qquad \chi^t \in \mathfrak{H}(\mathcal{B},\mathbb{E}_\phi), \tag{15}$$

for which $i(\chi^t) \in \mathfrak{H}(\mathcal{B},\mathbb{E}_{\phi^*})$, and the values of $\mathcal{F}_{\phi_i} \in \mathbb{V}_{\phi^*}\otimes\mathbb{V}_{\phi^*}$.

Now, consider another isometry between $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$,

$$j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}, \qquad j(\boldsymbol{x}) = \mathcal{J}(t)(\boldsymbol{x} - \boldsymbol{x}_j) + \boldsymbol{x}_j^*(t). \tag{16}$$

Similarly, there is a change of frame $\phi \to \phi_j$ with $\mathcal{J} \in \mathcal{O}(\mathbb{V}_\phi,\mathbb{V}_{\phi^*})$ taking $(\boldsymbol{x},t) \mapsto (j(\boldsymbol{x}),t)$, and $\mathbb{E}_{\phi_j} = \mathbb{E}_{\phi^*}$, and we have the condition of Euclidean objectivity,

$$\mathcal{F}_{\phi_j}(j(\chi^t);p) = \mathcal{J}(t)\,\mathcal{F}_\phi(\chi^t;p)\,\mathcal{J}(t)^\top, \qquad \chi^t \in \mathfrak{H}(\mathcal{B},\mathbb{E}_\phi), \tag{17}$$

for which $j(\chi^t) \in \mathfrak{H}(\mathcal{B},\mathbb{E}_{\phi^*})$, and the values of $\mathcal{F}_{\phi_j} \in \mathbb{V}_{\phi^*}\otimes\mathbb{V}_{\phi^*}$ as before.

Note that the two isometries induce two changes of frame from ϕ to two different frames ϕ_i and ϕ_j in the same Euclidean space $\mathbb{E}_{\phi^*}$. Moreover, the two constitutive functions $\mathcal{F}_{\phi_i}$ and $\mathcal{F}_{\phi_j}$ have the common domain $\mathfrak{H}(\mathcal{B},\mathbb{E}_{\phi^*})$ and their values are in the same space $\mathbb{V}_{\phi^*}\otimes\mathbb{V}_{\phi^*}$. Therefore, we can postulate:

Principle of material frame-indifference. *Let $\phi \to \phi_i$ and $\phi \to \phi_j$ be two changes of frame induced by two isometries $i,j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$, then the corresponding constitutive function $\mathcal{F}_{\phi_i}$ and $\mathcal{F}_{\phi_j}$ must have the same form,*

$$\mathcal{F}_{\phi_i}(\,\bullet\,;p) = \mathcal{F}_{\phi_j}(\,\bullet\,;p), \qquad p \in \mathcal{B}, \quad \bullet \in \mathfrak{H}(\mathcal{B},\mathbb{E}_{\phi^*}). \tag{18}$$

This simple relation renders mathematically the basic idea of frame-indifference of material behavior: the constitutive function, which models the intrinsic behavior of the material, is independent of observer, i.e., $\mathcal{F}_{\phi_i} = \mathcal{F}_{\phi_j}$ (equivalence of (13) when $\mathbb{E}_\phi = \mathbb{E}_{\phi^*}$).

We can now easily deduce the restriction on the constitutive function imposed by the principle of material frame-indifference. From the relation (15) and (17), we have

$$\begin{aligned}
\mathcal{F}_\phi(\chi^t;p) &= \mathcal{I}(t)^\top\,\mathcal{F}_{\phi_i}(i(\chi^t);p)\,\mathcal{I}(t) = \mathcal{I}(t)^\top\,\mathcal{F}_{\phi_j}(i(\chi^t);p)\,\mathcal{I}(t)\\
&= \mathcal{I}(t)^\top\,\mathcal{F}_{\phi_j}(j\circ(j^{-1}\circ i)(\chi^t);p)\,\mathcal{I}(t)\\
&= \mathcal{I}(t)^\top\mathcal{J}(t)\,\mathcal{F}_\phi(\underline{(j^{-1}\circ i)}(\chi^t);p)\,\mathcal{J}(t)^\top\mathcal{I}(t).
\end{aligned}$$

The underlined composite mapping $q = (j^{-1} \circ i)$ is a Euclidean transformation (an isometry) from $\mathbb{E}_\phi$ to itself,

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\boldsymbol{x}) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{r}(t), \tag{19}$$

where $Q = \mathcal{J}^\top \mathcal{I} \in O(\mathbb{V}_\phi)$ is an *orthogonal* transformation on $\mathbb{V}_\phi$, and some $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$. Indeed, from (16), we have for any $\boldsymbol{x}^* \in \mathbb{E}_{\phi^*}$,

$$j^{-1}(\boldsymbol{x}^*) = \boldsymbol{x} = \mathcal{J}^\top(t)(\boldsymbol{x}^* - \boldsymbol{x}_j^*(t)) + \boldsymbol{x}_j,$$

and hence with (14), the transformation (19) follows from

$$(j^{-1} \circ i)(\boldsymbol{x}) = \mathcal{J}^\top(t)\mathcal{I}(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \Big(\mathcal{J}^\top(t)\mathcal{I}(t)(\boldsymbol{x}_0 - \boldsymbol{x}_i) + \mathcal{J}^\top(t)(\boldsymbol{x}_i^*(t) - \boldsymbol{x}_j^*(t)) + \boldsymbol{x}_j\Big),$$

with $\boldsymbol{r}(t)$ being the terms in the big parentheses.

Therefore, from the above relations, we obtain the following consequence of the principle of material frame-indifference:

Condition of material objectivity. *In a frame of reference ϕ, the constitutive function $\mathcal{F}_\phi$ must satisfies the condition,*

$$\mathcal{F}_\phi(q(\mathcal{X}^t); p) = Q(t)\, \mathcal{F}_\phi(\mathcal{X}^t; p)\, Q(t)^\top, \qquad p \in \mathcal{B}, \tag{20}$$

for any history of motion $\mathcal{X}^t \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_\phi)$ and any Euclidean transformation

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\boldsymbol{x}) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{r}(t),$$

with some orthogonal transformation $Q(t) \in O(\mathbb{V}_\phi)$, and some $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$.

Since the condition (20) involves only one single frame of reference ϕ, it imposes a restriction on the constitutive function $\mathcal{F}_\phi$. Sometimes, the condition of material objectivity is referred to as the "principle of material objectivity", to impart its relevance in characterizing material property and Euclidean objectivity, as a more explicit form of the principle of material-frame indifference. Indeed, the original principle of material frame-indifference in the fundamental treatise by Truesdell and Noll [17] was formulated in the form (20) instead of more intuitive expression (18).

Remark 5. There is an apparent similarity between the two relations (12) and (20). We emphasize that in the condition of Euclidean objectivity (12), $\mathcal{Q}(t)$ is *the* orthogonal (isometric) transformation associated with the *change* of frame from ϕ to ϕ^*. While the condition of material objectivity (20) is valid in a *single* frame ϕ for some *arbitrary* orthogonal transformation $Q(t)$. Nevertheless, this apparent similarity still causes some confusions in the literature in the occasional use of the "principle of frame-indifference" ([2, 16, 18]). From our discussions so far, as we understand, the frame-indifference should not be regarded as a physical principle, it merely concerns the transformation properties due to changes of frame for kinematic or non-kinematic quantities, while the principle of material frame-indifference concerns material properties relative to different observers. □

8 Constitutive equations in referential description

For mathematical analysis, it is more convenient to use referential description so that motions can be defined on the Euclidean space instead of the set of material points. Therefore, for further discussions, we shall reinterpret the principle of material frame-indifference for constitutive equations, or equivalently the condition of material objectivity, relative to a reference configuration.

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Fig. 2), then $\kappa_\phi = \phi_{t_0} \circ \kappa : \mathcal{B} \to \mathbb{E}_\phi$ is the reference configurations of $\mathcal{B}$ in the frame ϕ, and

$$\boldsymbol{X} = \kappa_\phi(p) \in \mathbb{E}_\phi, \qquad p \in \mathcal{B}, \qquad \mathcal{B}_\kappa = \kappa_\phi(\mathcal{B}) \subset \mathbb{E}_\phi.$$

The motion $\chi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi$ relative to the reference configuration κ_ϕ is given by

$$\chi_\kappa(\cdot, t) : \mathcal{B}_\kappa \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi(p,t) = \chi\big(\kappa_\phi^{-1}(\boldsymbol{X}), t\big) = \chi_\kappa(\boldsymbol{X}, t), \qquad \chi = \chi_\kappa \circ \kappa_\phi.$$

We can define the corresponding constitutive functions with respect to the reference configuration,

$$\mathcal{F}_\phi(\chi^t; p) = \mathcal{F}_\phi(\chi_\kappa^t \circ \kappa_\phi;\, \kappa_\phi^{-1}(\boldsymbol{X})) := \mathcal{F}_\kappa(\chi_\kappa^t; \boldsymbol{X}),$$

and from (20), the condition of material objectivity for the constitutive function in the reference configuration can be restated as

$$\mathcal{F}_\kappa(q(\chi_\kappa^t); \boldsymbol{X}) = Q(t)\, \mathcal{F}_\kappa(\chi_\kappa^t; \boldsymbol{X})\, Q(t)^\top, \qquad \boldsymbol{X} \in \mathcal{B}_\kappa, \tag{21}$$

for any history of motion $\chi_\kappa^t \in \mathfrak{H}(\mathcal{B}_\kappa, \mathbb{E}_\phi)$ and any Euclidean transformation

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\chi_\kappa(\boldsymbol{X}, t)) = Q(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_0) + \boldsymbol{r}(t),$$

for some orthogonal transformation $Q(t) \in O(\mathbb{V}_\phi)$, and some $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$.

Remark 6. Note that the condition (21) is valid for any Euclidean transformation $q : \mathbb{E}_\phi \to \mathbb{E}_\phi$, which can also be interpreted as a time-dependent rigid deformation of the body in the Euclidean space $\mathbb{E}_\phi$. This interpretation is sometimes viewed as an alternative version of the principle of material frame-indifference and is called the "principle of invariance under superimposed rigid body motions". □

9 Simple materials

According to the principle of determinism (10), thermomechanical histories of any part of the body can affect the response at any point of the body. In most applications, such a non-local property is irrelevant. Therefore, it is usually assumed that only thermomechanical histories in an arbitrary small neighborhood of $\boldsymbol{X}$ affects the material response at the point $\boldsymbol{X}$, and hence the global history functions can be approximated at $\boldsymbol{X}$ by Taylor series up to certain order in a small neighborhood of $\boldsymbol{X}$. In particular, when only linear approximation is concerned, the constitutive function is restricted to a special class of materials,

$$\mathcal{F}_\kappa(\chi_\kappa^t(\cdot), \boldsymbol{X}) = \mathcal{H}_\kappa(\nabla_{\boldsymbol{X}} \chi_\kappa^t(\boldsymbol{X}), \boldsymbol{X}),$$

so that we can write the constitutive equation for the stress as

$$T(\boldsymbol{X}, t) = \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X}), \qquad F_\kappa^t \in \mathfrak{H}(\{\boldsymbol{X}\}, L(\mathbb{V}_\phi)), \qquad \boldsymbol{X} \in \mathcal{B}_\kappa, \tag{22}$$

where $F_\kappa^t = \nabla_{\boldsymbol{X}} \chi_\kappa^t$ is the deformation gradient and the domain of the history is a single point $\{\boldsymbol{X}\}$. Note that although the constitutive function depends only on local values at the position $\boldsymbol{X}$, it is still general enough to define a material with memory of local deformation in the past. A material with constitutive equation (22) is called a *simple material* by Noll. The class of simple materials is general enough to include most of the materials of practical interests, such as: elastic solids, viscoelastic solids, as well as Navier-Stokes fluids and non-Newtonian fluids.

For the Euclidean transformation $q : \mathbb{E}_\phi \to \mathbb{E}_\phi$ and $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ from (21), we have

$$\nabla_{\boldsymbol{X}} q(\chi_\kappa^t(\boldsymbol{X})) = \nabla_{\boldsymbol{x}} q(\chi_\kappa^t(\boldsymbol{X})) \nabla_{\boldsymbol{X}} \chi_\kappa^t(\boldsymbol{X}) = Q^t F_\kappa^t(\boldsymbol{X}).$$

Therefore, we obtain the following main result for simple materials:

Condition of material objectivity. *For simple materials relative to a reference configuration, the constitutive equation* $T(\boldsymbol{X}, t) = \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X})$ *satisfies*

$$\mathcal{H}_\kappa\big(Q^t F_\kappa^t; \boldsymbol{X}\big) = Q(t)\, \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X})\, Q(t)^\top, \tag{23}$$

for any history of deformation gradient $F^t_\kappa \in \mathfrak{H}(\{\boldsymbol{X}\}, L(\mathbb{V}_\phi))$ and any orthogonal transformation $Q(t) \in O(\mathbb{V}_\phi)$.

Remark 7. The condition (23) is the most well-known result in constitutive theories of continuum mechanics. It is the ultimate goal to obtain this result regardless of whoever agree or disagree with each other on the formulation and interpretation of frame-indifference and the principle of material frame-indifference. □

It is also interesting to see how the principle of material frame-indifference in the form (18), i.e., $\mathcal{F}_{\phi_i} = \mathcal{F}_{\phi_j}$, takes in referential description. Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0, and $i : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$ be an isometry, then

$$\kappa_i : \mathcal{B} \to \mathbb{E}_{\phi^*}, \qquad \kappa_i = i \circ \kappa_\phi = i \circ \phi_{t_0} \circ \kappa,$$

is the reference configurations of $\mathcal{B}$ in the frame $\phi_i = i \circ \phi_{t_0}$ under the isometry i, and

$$\boldsymbol{X}_i = \kappa_i(p) \in \mathbb{E}_{\phi^*}, \qquad p \in \mathcal{B}, \qquad \mathcal{B}_{\kappa_i} = \kappa_i(\mathcal{B}) \subset \mathbb{E}_{\phi^*}.$$

The motion $\chi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi$ relative to the reference configuration κ_i is given by

$$\chi_{\kappa_i}(\cdot, t) : \mathcal{B}_{\kappa_i} \to \mathbb{E}_{\phi^*}, \qquad \boldsymbol{x}_i = \chi_{\phi_i}(p, t) = i(\chi(p, t)) = \chi_{\kappa_i}(\kappa_i(p), t),$$

so that we have

$$\chi_{\phi_i} = i \circ \chi = \chi_{\kappa_i} \circ \kappa_i, \qquad \chi_{\kappa_i} = i \circ \chi \circ \kappa_i^{-1}. \tag{24}$$

We can define the constitutive functions with respect to the reference configuration κ_i,

$$\mathcal{F}_{\phi_i}(i(\chi^t); p) = \mathcal{F}_{\phi_i}(\chi^t_{\kappa_i} \circ \kappa_i;\, \kappa_i^{-1}(\boldsymbol{X}_i)) := \mathcal{F}_{\kappa_i}(\chi^t_{\kappa_i}; \boldsymbol{X}_i),$$

and we can do similar things for another isometry $j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$. Then from (18), after some calculations, we obtain

$$\mathcal{F}_{\kappa_i}(\chi^t_{\kappa_j} \circ \kappa_j \circ \kappa_i^{-1};\, \boldsymbol{X}_i) = \mathcal{F}_{\kappa_j}(\chi^t_{\kappa_j}; \boldsymbol{X}_j), \tag{25}$$

for any deformation history $\chi^t_{\kappa_j} \in \mathfrak{H}(\mathcal{B}_{\kappa_j}, \mathbb{E}_{\phi^*})$ and from (24), we have

$$\chi_{\kappa_j} = q^* \circ \chi_{\kappa_i} \circ \kappa_i \circ \kappa_j^{-1},$$

where $q^* = j \circ i^{-1} : \mathbb{E}_{\phi^*} \to \mathbb{E}_{\phi^*}$ is an isometry with the associated orthogonal transformation $Q^* = \mathcal{J}\mathcal{I}^\top \in O(\mathbb{V}_{\phi^*})$.

By the use of the above relation and (25), from (15) and (17), we obtain

$$\mathcal{F}_{\kappa_i}(q^*(\chi^t_{\kappa_i}); \boldsymbol{X}_i) = Q^*(t)\mathcal{F}_{\kappa_i}(\chi^t_{\kappa_i}; \boldsymbol{X}_i)Q^*(t)^\top, \qquad \boldsymbol{X}_i \in \mathcal{B}_{\kappa_i}, \tag{26}$$

for any deformation history $\chi_{\kappa_i} \in \mathfrak{H}(\mathcal{B}_{\kappa_i}, \mathbb{E}_{\phi^*})$ and Euclidean transformation $q^* : \mathbb{E}_{\phi^*} \to \mathbb{E}_{\phi^*}$ with orthogonal transformation $Q^*(t) \in O(\mathbb{V}_{\phi^*})$, whcih can now be chosen arbitrarily because the relation (26) referred only to one frame of reference only.

Of course, we can see that the above condition (26) formulated in the Euclidean space $\mathbb{E}_{\phi^*}$ is equivalent to the condition of material objectivity (21) derived directly from (20) in $\mathbb{E}_\phi$.

Remark 8. In the principle of material frame-indifference, we have postulated that constitutive functions are independent of observers stated as $\mathcal{F}_{\phi_i} = \mathcal{F}_{\phi_j}$ in (18). We emphasize that this is valid only when it is formulated in material description. Indeed, from the above discussion, we have $\mathcal{F}_{\kappa_i} \neq \mathcal{F}_{\kappa_j}$ in referential description, instead, they must satisfy the relation (25). □

10 Some general remarks

Remark 9. It is interesting to give the following example, which shows typically why some misconception persisted. Let F_κ, F^*_κ and $T = \mathcal{T}_\kappa(F_\kappa)$, $T^* = \mathcal{T}^*_\kappa(F^*_\kappa)$, be the deformation gradients and the constitutive equations for the stress in two different frames relative to some reference configuration κ. One finds, in most textbooks, that the objectivity conditions are given by

$$F^*_\kappa = QF_\kappa, \qquad \mathcal{T}^*_\kappa(F^*_\kappa) = Q\,\mathcal{T}_\kappa(F_\kappa)\,Q^\top,$$

and the principle of material frame-indifference by

$$\mathcal{T}^*_\kappa(\bullet) = \mathcal{T}_\kappa(\bullet),$$

which combine to give the well-known condition of material objectivity,

$$\mathcal{T}_\kappa(QF_\kappa) = Q\,\mathcal{T}_\kappa(F_\kappa)\,Q^\top.$$

This is the correct result equivalent to the condition (23). However, we have already shown that $F^*_\kappa = QF_\kappa$ is valid only when the reference configuration is unaffected by the change of frame, and the principle of material frame-indifference does not imply $\mathcal{T}^*_\kappa(\bullet) = \mathcal{T}_\kappa(\bullet)$ in referential description. Nevertheless, the lucky incident that the two inadequate assumptions would lead to the correct general result is quite striking and might have contributed to some misconception over the decades. □

Remark 10. To discuss the notion of invariance under observer changes in Sec. 20.1 of [2], it distinguishes the observed space and the reference space, the former being the ambient space through which the body evolves and the later being the body at the reference configuration. And it emphasizes that *while a change of frame affects the observed space through which the deformed body evolves, it does not affect the reference space*. In order to emphasize this concept, the transformation of relative deformation gradient under the change of frame is given as an example. The relative deformation gradient,

$$F_{(t)}(\tau) = F(\tau)F(t)^{-1},$$

is derived using the configuration at time t as reference, hence [it states that] $F(t)$ is unaffected by the change of frame. Therefore, the transformation law for $F_{(t)}(\tau)$ should be the same as that for $F(\tau)$, namely (p. 150, [2]),

$$F^*_{(t)}(\tau) = Q(\tau)F_{(t)}(\tau), \tag{27}$$

instead of the usual one (see Sec. 29 of [17] and elsewhere),

$$F^*_{(t)}(\tau) = Q(\tau)F_{(t)}(\tau)Q(t)^\top. \tag{28}$$

To show that (27) is incorrect, let us consider simple materials. Since

$$F(X, t-s) = F_{(t)}(X, t-s)F(X, t),$$

we can write the constitutive function in general as

$$T(X,t) = \mathcal{T}(F^t(X,s)) := \mathcal{F}(F^t_{(t)}(X,s), F(X,t)).$$

For simple fluids, one can show that the dependence of F reduces to the dependence of the density[1],

$$T(X,t) = \mathcal{F}(F^t_{(t)}(X,s), \rho(X,t)),$$

[1] The properties of material symmetry has not been treated in this paper, because the concept of material symmetry transformation concerns only changes of reference configuration in a fixed frame of reference, therefore, no changes of frame are involved.

and the material objectivity condition,

$$\mathcal{F}(F^{t*}_{(t)}, \rho) = Q(t)\mathcal{F}(F^t_{(t)}, \rho)Q(t)^\top,$$

can be written in the following form if the relation (27) is correct,

$$\mathcal{F}(Q^t F^t_{(t)}, \rho) = Q(t)\mathcal{F}(F^t_{(t)}, \rho)Q(t)^\top. \tag{29}$$

Since this condition must hold for any $Q(t) \in O(\mathbb{V})$, let $F^t_{(t)} = R^t_{(t)}U^t_{(t)}$ be the polar decomposition, and with $Q^t = (R^t_{(t)})^\top$, the condition (29) implies that

$$\mathcal{F}(F^t_{(t)}, \rho) = \mathcal{F}(U^t_{(t)}, \rho),$$

since $Q(t) = (R_{(t)}(t))^\top = I$. In other words, the dependence of the function $\mathcal{F}$ on $F^t_{(t)}$ reduces to only its right stretch part $U^t_{(t)}$.

Now return to the condition (29), for the argument on the left hand side, since

$$Q^t F^t_{(t)} = (Q\, R^t_{(t)})U^t_{(t)},$$

which by uniqueness of polar decomposition, the right stretch part of $Q^t F^t_{(t)}$ is $U^t_{(t)}$. Therefore, the condition (29) becomes

$$\mathcal{F}(U^t_{(t)}, \rho) = Q(t)\mathcal{F}(U^t_{(t)}, \rho)Q(t)^\top \qquad \forall\, Q(t) \in O(\mathbb{V}), \tag{30}$$

which implies that $\mathcal{F}Q = Q\mathcal{F}$ for any orthogonal transformation. Therefore, the constitutive function $\mathcal{F}$ must reduce to a function proportional to the identity tensor. *This conclusion is obviously absurd for simple fluids in general.* This proves that the relation (27) cannot be correct.

On the other hand, to get the correct result with the use of the correct transformation law (28), one can prove, instead of (30), the following condition (see for example, [3, 17]),

$$\mathcal{F}(Q(t)U^t_{(t)}Q(t)^\top, \rho) = Q(t)\mathcal{F}(U^t_{(t)}, \rho)Q(t)^\top \qquad \forall\, Q(t) \in O(\mathbb{V}),$$

for the reduced constitutive equation $T(X,t) = \mathcal{F}(U^t_{(t)}(X,s), \rho(X,t))$ of simple fluids in general. □

Remark 11. On frame-free theory:

In [13], Noll pointed out that it should be possible to make the principle of material frame-indifference vacuously satisfied by a frame-free formulation in describing internal interactions of a physical system. We would like to take a look at such a theory.

Setting aside the somehow esoteric mathematical structure and notations used by Noll (interested reader can consult [14]), we shall try to examine the intricate definitions for its underlaying assumptions, using our conventional notations, hopefully without loss of their essential meanings.

First of all, we have to clarify some terminology. Without using a frame of reference, he defines a body $\mathcal{B}$ as a three-dimensional differentiable manifold and a mapping $\kappa : \mathcal{B} \to \mathbb{E}$, from the body into a Euclidean space $\mathbb{E}$, is called a *placement* (what we call configuration). The mapping $\gamma \circ \kappa^{-1}$, from one placement κ to another γ, is called *transplacement* (what we call deformation).

A placement κ induces a metric on the body, $d_\kappa(X,Y) = |\kappa(X) - \kappa(Y)|$ for all $X, Y \in \mathcal{B}$. He calls this metric the *configuration* induced by the placement κ (not to be confused with what we call configuration). By this definition, any two placements differ by an isometric transplacement, such as a rigid body motion, will give rise to the same configuration.

The *intrinsic* stress S is defined as a symmetric tensor on the tangent space of the body manifold. It is then postulated that a frame-free constitutive law should involve *only* such intrinsic stresses [14] and for an elastic material element, the response function of the intrinsic stress is defined in the following form (Eqn. (2.6) of [13]),

$$\mathrm{S} = \mathfrak{h}(\mathrm{G}), \tag{31}$$

where G is called the *configuration* of the tangent space induced by the configuration d_κ of the placement κ. Such a configuration is represented by the inner product induced by the positive-definite bilinear function $K^\top K$ on the tangent space $\mathcal{T}_X$ at $X \in \mathcal{B}$, where $K = \nabla_X \kappa : \mathcal{T}_X \to \mathbb{V}$ is the placement gradient, where $\mathbb{V}$ is the translation space of the Euclidean space $\mathbb{E}$. (it is essentially the second order tensor $F^\top F$ in our notation with F being the deformation gradient). Therefore, given another placement $\bar\kappa$, such that $\bar K = \nabla_X \bar\kappa = QK$ for some $Q \in \mathrm{Orth}(\mathbb{V})$, it follows that $\bar K^\top \bar K = K^\top K$, and hence it gives rise to the same configuration G. — In other words, the constitutive equation (31) is invariant under superposed rigid body motions by definition.

The intrinsic stress is defined as $\mathrm{S} = F^{-1} T F^{-\top}$, where T is the Cauchy stress. For elastic materials, it is well-known that the reduced constitutive equation, which satisfies the principle of material frame-indifference identically, can be expressed in terms of the second Piola-Kirchhoff stress tensor, $\tilde{\mathrm{T}} = (\det F)\mathrm{S}$, in the form (it follows from the condition (23), also see [17, 3] etc.),

$$\tilde{\mathrm{T}} = \sqrt{\det C}\, \mathrm{S} = \mathfrak{t}(C),$$

where $C = F^T F$ is the right Cauchy–Green strain tensor, or essentially $\mathrm{S} = \mathfrak{h}(C)$. — In other words, the frame-free theory conveniently takes the well-known result from the principle of material frame-indifference as axiomatic definition of intrinsic response function (31).

Surely, it is remarkable to have an interesting frame-free theory of Noll. However, from the above observations, the claim that such a formulation makes the principle of material frame-indifference vacuous is largely misleading. Because it essentially employs the well-known consequences of the principle in its axiomatic definition of *configuration* as the metric of placement and the response function for the conveniently chosen *intrinsic stress*. Therefore, it seems that the premises of the frame-free theory of elasticity [13], are built on the basic ideas of material frame-indifference (or equivalently, invariant under superimposed rigid body motions) as the hidden foundation. □

Remark 12. The first controversy on MFI was raised by Müller in 1972 [9] that kinetic theory of gases does not support the principle of material frame-indifference, according to which constitutive functions must be frame-independent. In that article, an iterative scheme akin to the Maxwellian iteration from moment equations in the kinetic theory, initiated with a certain equilibrium state, is employed to obtain the first and the second iterates for stress and heat flux in terms of basic fields of continuum mechanics, namely, fields of density, motion and temperature. These relations are then regarded as constitutive equations. Indeed, the first iterates yield Navier–Stokes and Fourier laws and unfortunately (or surprisingly), the much elaborated second iterates contain terms depending on the rotation of the frame — a clear violation of MFI in continuum mechanics. This controversy set off spontaneous debates on the validity of MFI and all-round discussions on the concept of frame-indifference ever since (see references in [1] and elsewhere).

In 1983, after the formulation of extended thermodynamics [7] for extended basic fields including stress and heat flux, in the framework of continuum mechanics (including MFI with Euclidean transformations), Müller sees "the violation of MFI in a new light" — It only reflects the frame dependence of the basic equations of balance (for stress and heat flux), while the theory is frame independent in the constitutive relations [*sic*]. In other words, there is no violation of MFI in the framework of continuum mechanics.

Indeed, this reminds us of the earlier discussion on Galilean invariance of Newton's law in Sec. 5. If we regard Newton's second law as a constitutive equation for the force in terms of motion, then in a non-inertial frame it is frame-dependent containing the rotation of the frame, because Newton's second law is Galilean invariant only. However, this is not what we would have done, instead, the force is postulated as a frame-indifferent vector quantity and the frame-dependent terms are recognized as inertial forces (Coriolis force for example) in a non-inertial frame. We can now draw the analogy with the balance equation for stress (in kinetic theory or extended thermodynamics), that the balance equation for the stress is Galilean invariant only and hence frame-dependent, while the stress is a frame-indifferent tensor quantity. Those controversial iterates are merely approximations of the balance equation itself, therefore containing frame-dependent terms in a non-inertial frame, which may contain contribution from Coriolis force due to rotation of the frame. In other words, regarding the Maxwellian iterates in kinetic theory of gases as constitutive equations in the sense of continuum mechanics, instead of as approximations of higher order moment equations of balance near certain equilibrium state, is the source of confusion in this controversy. □

Appendix: Isometries of Euclidean spaces

For a Euclidean space $\mathbb{E}$, there is a vector space $\mathbb{V}$, called the translation space of $\mathbb{E}$, such that the difference $\boldsymbol{v} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ of any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ is a vector in $\mathbb{V}$. We also require that the vector space $\mathbb{V}$ be equipped with an inner product $(\cdot,\cdot)_{\mathbb{V}}$, so that length and angle can be defined.

Let $\mathbb{V}^*$ be the translation space of another Euclidean space $\mathbb{E}^*$ and $\mathcal{L}(\mathbb{V},\mathbb{V}^*)$ be the space of linear transformations from $\mathbb{V}$ to $\mathbb{V}^*$. For $A \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$, the transpose (or more generally called adjoint) of A, denoted by $A^\top \in \mathcal{L}(\mathbb{V}^*,\mathbb{V})$ satisfies $(\boldsymbol{u}, A^\top \boldsymbol{v}^*)_{\mathbb{V}} = (A\boldsymbol{u}, \boldsymbol{v}^*)_{\mathbb{V}^*}$ for $\boldsymbol{u} \in \mathbb{V}$ and $\boldsymbol{v}^* \in \mathbb{V}^*$.

Remark. In an inner product space, since the norm is defined as $\|\boldsymbol{u}\|_{\mathbb{V}} = \sqrt{(\boldsymbol{u},\boldsymbol{u})_{\mathbb{V}}}$, from the identity, $\|\boldsymbol{u}+\boldsymbol{v}\|^2_{\mathbb{V}} = \|\boldsymbol{u}\|^2_{\mathbb{V}} + \|\boldsymbol{v}\|^2_{\mathbb{V}} + 2(\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}}$, it follow that

$$\|\boldsymbol{u}\|^2_{\mathbb{V}} = \|\boldsymbol{u}^*\|^2_{\mathbb{V}^*} \iff (\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}} = (\boldsymbol{u}^*,\boldsymbol{v}^*)_{\mathbb{V}^*},$$

for any corresponding $\boldsymbol{u},\boldsymbol{v} \in \mathbb{V}$ and $\boldsymbol{u}^*,\boldsymbol{v}^* \in \mathbb{V}^*$. □

Definition. *$\mathcal{I} \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$ is called an isometric transformation if $\|\mathcal{I}\boldsymbol{u}\|_{\mathbb{V}^*} = \|\boldsymbol{u}\|_{\mathbb{V}}$ for any $\boldsymbol{u} \in \mathbb{V}$. Let $\mathcal{O}(\mathbb{V},\mathbb{V}^*)$ denote the set of all isometric transformations in $\mathcal{L}(\mathbb{V},\mathbb{V}^*)$.*

From the above observation, an isometric transformation preserves the norm as well as the inner product, the length and the angle.

Definition. *(Isometry): A bijective map $i : \mathbb{E} \to \mathbb{E}^*$ is an isometry if for $\boldsymbol{x} \in \mathbb{E}$,*

$$\boldsymbol{x}^* = i(\boldsymbol{x}) = \mathcal{I}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{x}^*_0, \tag{1}$$

for some $\boldsymbol{x}_0 \in \mathbb{E}$, $\boldsymbol{x}^*_0 \in \mathbb{E}^*$ and some $\mathcal{I} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$.

Let $L(\mathbb{V}) = \mathcal{L}(\mathbb{V},\mathbb{V})$ be the space of linear transformations and $O(\mathbb{V}) = \mathcal{O}(\mathbb{V},\mathbb{V})$ be the group of orthogonal transformation on $\mathbb{V}$. Note that $\mathcal{O}(\mathbb{V},\mathbb{V}^*)$ does not have a group structure in general. The transformation (1) is often referred to as a *Euclidean* transformation when $\mathbb{E} = \mathbb{E}^*$ and $\mathbb{V} = \mathbb{V}^*$. In this case $\mathcal{I} \in O(\mathbb{V})$ is an orthogonal transformation.

For $\mathcal{I} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$, it follows that $\mathcal{I}^\top\mathcal{I} = I_{\mathbb{V}}$ and $\mathcal{I}\mathcal{I}^\top = I_{\mathbb{V}^*}$ are identity transformations. Hence $\mathcal{I}^\top = \mathcal{I}^{-1}$ and $\mathcal{I}^\top \in \mathcal{O}(\mathbb{V}^*,\mathbb{V})$ is an isometric transformation from $\mathbb{V}^*$ to $\mathbb{V}$. Moreover, if $\mathcal{R} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$, then $\mathcal{I}^\top\mathcal{R} \in O(\mathbb{V})$ and $\mathcal{I}\mathcal{R}^\top \in O(\mathbb{V}^*)$ are orthogonal transformations on $\mathbb{V}$ and $\mathbb{V}^*$ respectively. Indeed, isometric transformation is the counterpart of orthogonal transformation when two different vector spaces are involved.

Note that for clarity, we have denoted transformations within the same space by capital letters, like $Q \in O(\mathbb{V}) \subset L(\mathbb{V})$, and transformations between two different spaces by script capital letters, like $\mathcal{Q} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*) \subset \mathcal{L}(\mathbb{V},\mathbb{V}^*)$.

Examples in $\mathbb{R}^2$:

Let $\mathbb{E} = \mathbb{R}^2$ and $\mathbb{V} = \mathbb{R}^2$ be its translation space with the standard inner product

$$(\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}} = u_1v_1 + u_2v_2, \quad \text{for} \quad \boldsymbol{u} = (u_1,u_2), \ \ \boldsymbol{v} = (v_1,v_2),$$

and let $\mathbb{E}^* = \mathbb{R}^2$ and $\mathbb{V}^* = \mathbb{R}^2$ be its translation space with the inner product given by

$$(\boldsymbol{u}^*,\boldsymbol{v}^*)_{\mathbb{V}^*} = a^2u^*_1v^*_1 + b^2u^*_2v^*_2, \quad \text{for} \quad \boldsymbol{u}^* = (u^*_1,u^*_2), \ \ \boldsymbol{v}^* = (v^*_1,v^*_2),$$

for some constant $a > 0$ and $b > 0$, which means it has different scaling in x- and y-directions.

Let the linear transformations R and R^* be given in matrix form relative to the standard basis $\beta = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$ by

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad [R^*] = \begin{bmatrix} \cos\theta & \dfrac{b}{a}\sin\theta \\ -\dfrac{a}{b}\sin\theta & \cos\theta \end{bmatrix}. \tag{2}$$

Obviously, the transformation R is orthogonal on $\mathbb{V}$, $R \in O(\mathbb{V})$, while one can show that R^* is orthogonal on $\mathbb{V}^*$, $R^* \in O(\mathbb{V}^*)$. Indeed, for $\boldsymbol{u}^* = (x^*, y^*)$ and $\hat{\boldsymbol{u}}^* = R^*\boldsymbol{u}^* = (\hat{x}^*, \hat{y}^*)$, we have

$$\hat{x}^* = x^*\cos\theta + \frac{b}{a}y^*\sin\theta, \qquad \hat{y}^* = -\frac{a}{b}x^*\sin\theta + y^*\cos\theta,$$

and hence

$$\|R^*\boldsymbol{u}^*\|_{\mathbb{V}^*} = a^2(\hat{x}^*)^2 + b^2(\hat{y}^*)^2 = a^2(x^*)^2 + b^2(x^*)^2 = \|\boldsymbol{u}^*\|_{\mathbb{V}^*}.$$

Now consider a linear transformation $\mathcal{S} : \mathbb{V} \to \mathbb{V}^*$ given by

$$[\mathcal{S}] = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}, \tag{3}$$

in matrix form relative to the stand basis. One can easily show that it is an isometric transformation, $\mathcal{S} \in \mathcal{O}(\mathbb{V}, \mathbb{V}^*)$, i.e., $\|\mathcal{S}\boldsymbol{u}\|_{\mathbb{V}^*} = \|\boldsymbol{u}\|_{\mathbb{V}}$.

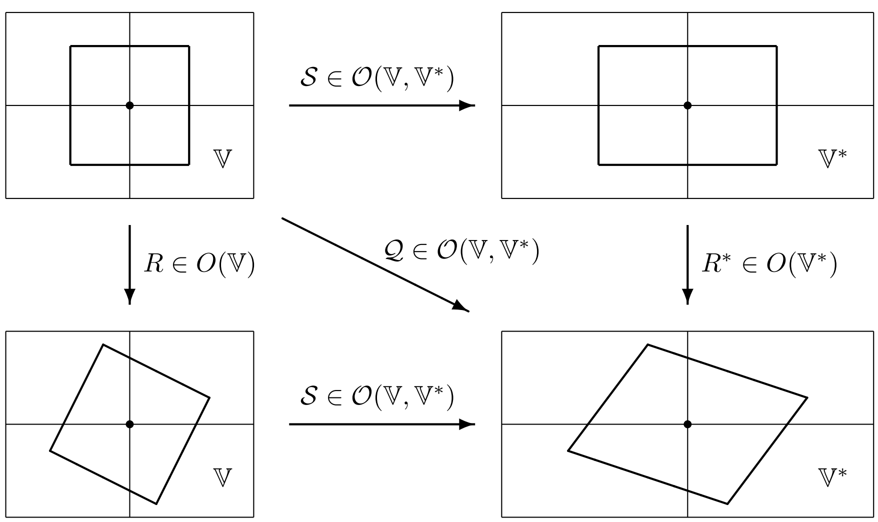

Figure 4: The effect of scaling and rotation: The views of the observer in space $\mathbb{E}$ on the left and in space $\mathbb{E}^*$ on the right. In this example, $\theta = \tan^{-1}(1/2)$ and $a = 2/3$, $b = 1$.

Geometrically, this transformation maps a figure in $\mathbb{V}$ to a similar figure scaled down/up in two different directions (x and y) in $\mathbb{V}^*$. This is like drawings of floor plan or street maps, in which the corresponding segments have the same physical length but in different scale, or in two different snapshots of the same figure from different distances.

We can also consider the composite maps of (2) and (3), and by direct computation, show that

$$\mathcal{Q} = \mathcal{S}R = R^*\mathcal{S},$$

which is a linear transformation from $\mathbb{V}$ to $\mathbb{V}^*$ given by (see Fig. 4)

$$[\mathcal{Q}] = \begin{bmatrix} \frac{1}{a}\cos\theta & \frac{1}{a}\sin\theta \\ -\frac{1}{b}\sin\theta & \frac{1}{b}\cos\theta \end{bmatrix}.$$

One can show that $\mathcal{Q} \in \mathcal{O}(\mathbb{V}, \mathbb{V}^*)$ is an isometric transformation, i.e., $\|\mathcal{Q}\boldsymbol{u}\|_{\mathbb{V}^*} = \|\boldsymbol{u}\|_{\mathbb{V}}$.

In Fig. 4 for an illustration, we show the image of a square as seen by a viewer in the space $\mathbb{E}$ and its image as seen by another viewer in the space $\mathbb{E}^*$. On the left hand side, the $\mathbb{E}$-viewer is seeing the square in a screen with aspect ratio of 4:3 as in standard television, while on the right hand side, the $\mathbb{E}^*$-viewer is seeing the stretched object in the wide-screen with aspect ratio 6:3. Note that on the top, the square is stretched into a rectangle, while at the bottom, the image of the rotated square is not even a rectangle. As a little surprise, the transformation R^* looks like a rotation and a shearing, even though we have shown that R^* is an orthogonal transformation in $O(\mathbb{V}^*)$ due to the uneven scaling in the definition of inner product of $\mathbb{V}^*$. Similarly, the isometric mapping $\mathcal{Q}$ transforms a square in $\mathbb{V}$ into a tilted parallelogram in $\mathbb{V}^*$ after a rotation R followed by a stretching $\mathcal{S}$, or a stretching $\mathcal{S}$ followed by a rotation R^*.

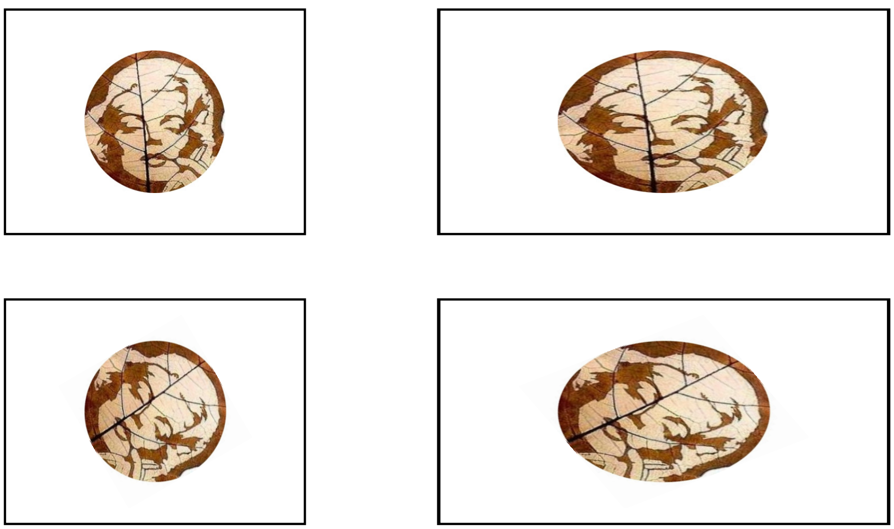

Figure 5: What one sees when switching from the standard 4:3 screen to the wide-screen mode of a television

In analogy with the above example, in Fig. 5 by switching from the standard 4:3 screen to the wide-screen mode in a television, one may see the change in images that not only stretches the human face horizontally, but also distorts the face by shearing when the head is tilted. Nevertheless in reality, both images belong to the same face as seen from different modes (viewing spaces).

Therefore, in consideration of change of observer, besides relative motion, it would be meaningful to allow relative orientation as well as scaling between observers, i.e., consenting observers may belong to different Euclidean spaces, related by isometric transformations.

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