A Continuum Mechanics Primer

On Constitutive Theories of Materials

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Preface

In this note, we concern only fundamental concepts of continuum mechanics for the formulation of basic equations of material bodies. Particular emphases are placed on general physical requirements, which have to be satisfied by constitutive equations of material models.

After introduction of kinematics for finite deformations and balance laws, constitutive relations for material bodies are discussed. Two general physical requirements for constitutive functions, namely, the principle of material frame-indifference and the material symmetry, are introduced and their general consequences analyzed. In particular, concepts of change of frame, Euclidean objectivity and observer-independence of material properties are carefully defined so as to make the essential ideas of the principle of material frame-indifference clear.

Constitutive equations and governing equations for some special classes of materials are then derived. For fluids, these includes elastic fluids, Navier-Stokes fluids, Reiner-Rivlin fluids, viscous heat-conducting fluids and incompressible fluids. For solids, are discussed isotropic elastic solids, anisotropic elastic solids, neo-Hookean material, Mooney-Rivlin materials for finite deformations in general as well as linearizations for small deformations, the Hookes' law in linear elasticity and the basic equations of linear thermoelasticity.

The constitutive theories of materials cannot be complete without some thermodynamic considerations, which like the conditions of material objectivity and material symmetry are equally important. Exploitation of the entropy principle based on the general entropy inequality and the stability of equilibrium are considered. The use of Lagrange multipliers in the evaluation of thermodynamic restrictions on the constitutive functions is introduced and its consequences for isotropic elastic solids are carefully analyzed.

This note can be regarded as a simplified and revised version of the book by the author (*Continuum Mechanics*, Springer 2002). It has been used in a short course for mathematics, physics and engineering students interested in acquiring a better knowledge of material modelling in continuum mechanics.

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Contents

CHAPTER 1

Kinematics

Notations

The reader is assumed to have a reasonable knowledge of the basic notions of vector spaces and calculus on Euclidean spaces. Some notations used in this note are introduced here. A concise treatment of tensor analysis as a preliminary mathematical background for continuum mechanics can be found in the appendix of [14].

Let V be a finite dimensional vector space with an inner product, and $\mathcal{L}(V)$ be the space of linear transformations on V. The elements of $\mathcal{L}(V)$ are also called (second order) tensors. For $\boldsymbol{u}$, $\boldsymbol{v} \in V$ their inner product is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, and their *tensor product*, denoted by $\boldsymbol{u} \otimes \boldsymbol{v}$, is defined as a tensor so that for any $\boldsymbol{w} \in V$,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\, \boldsymbol{u}.$$

Let $\{\boldsymbol{e}_i,\ i = 1, \cdots, n\}$ be a basis of V, then $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j,\ i, j = 1, \cdots, n\}$ is a basis for $\mathcal{L}(V)$, and for any $T \in \mathcal{L}(V)$, the component form can be expressed as

$$T = T^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} T^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

Hereafter, we shall always use the *summation convention* for any pair of repeated indices, as shown in the above example, for convenience and clarity. Moreover, although sometimes the super- and sub-indices are deliberately used to indicate the contravariant and the covariant components respectively (and sum over a pair of repeated super- and sub-indices only), in most theoretical discussions, expressions in Cartesian components suffice, for which no distinction of super- and sub-indices is necessary.

The inner product of two tensors A and B is defined as

$$A \cdot B = \operatorname{tr} AB^T,$$

where the trace and the transpose of a tensor are involved. In particular in terms of Cartesian components, the norm $|A|$ is given by

$$|A|^2 = A \cdot A = A_{ij} A_{ij},$$

which is the sum of square of all the elements of A by the summation convention.

The *Kronecker delta* and the *permutation symbol* are two frequently used notations, they are defined as

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases}$$

and

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1, & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ 0, & \text{if otherwise.} \end{cases}$$

One can easily check the following identity:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

Let $\mathcal{E}$ be a three-dimensional Euclidean space and the vector space V be its *translation space*. For any two points $\boldsymbol{x}$, $\boldsymbol{y} \in \mathcal{E}$ there is a unique vector $\boldsymbol{v} \in V$ associated with their difference,

$$\boldsymbol{v} = \boldsymbol{y} - \boldsymbol{x}, \quad \text{or} \quad \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{v}.$$

We may think of $\boldsymbol{v}$ as the geometric vector that starts at the point $\boldsymbol{x}$ and ends at the point $\boldsymbol{y}$. The *distance* between $\boldsymbol{x}$ and $\boldsymbol{y}$ is then given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}| = |\boldsymbol{v}|.$$

Let $\mathcal{D}$ be an open region in $\mathcal{E}$ and W be any vector space or an Euclidean space. A function $f : \mathcal{D} \to W$ is said to be *differentiable* at $\boldsymbol{x} \in \mathcal{D}$ if there exists a linear transformation $\nabla f(\boldsymbol{x}) : V \to W$, such that for any $\boldsymbol{v} \in V$,

$$f(\boldsymbol{x} + \boldsymbol{v}) - f(\boldsymbol{x}) = \nabla f(\boldsymbol{x})[\boldsymbol{v}] + o(\boldsymbol{v}),$$

where $o(\boldsymbol{v})$ denotes the higher order terms in $\boldsymbol{v}$ such that

$$\lim_{|\boldsymbol{v}| \to 0} \frac{o(\boldsymbol{v})}{|\boldsymbol{v}|} = 0.$$

We call ∇f the *gradient* of f with respect to $\boldsymbol{x}$, and will also denote it by $\nabla_{\boldsymbol{x}} f$, or more frequently by either $\operatorname{grad} f$ or $\operatorname{Grad} f$.

1.1 Configuration and deformation

A body $\mathcal{B}$ can be identified mathematically with a region in a three-dimensional Euclidean space $\mathcal{E}$. Such an identification is called a configuration of the body, in other words, a one-to-one mapping from $\mathcal{B}$ into $\mathcal{E}$ is called a *configuration* of $\mathcal{B}$.

It is more convenient to single out a particular configuration of $\mathcal{B}$, say κ, as a reference,

$$\kappa : \mathcal{B} \to \mathcal{E}, \qquad \kappa(\boldsymbol{p}) = \boldsymbol{X}. \tag{1.1}$$

We call κ a *reference configuration* of $\mathcal{B}$. The coordinates of $\boldsymbol{X}$, $(X^\alpha, \alpha = 1, 2, 3)$ are called the *referential coordinates*, or sometimes called the *material coordinates* since the point $\boldsymbol{X}$ in the reference configuration κ is often identified with the material point $\boldsymbol{p}$ of the body when κ is given and fixed. The body $\mathcal{B}$ in the configuration κ will be denoted by $\mathcal{B}_\kappa$.

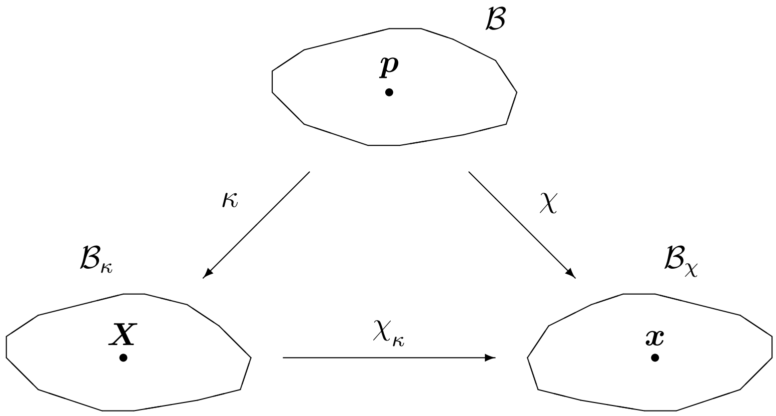

Figure 1.1: Deformation

Let κ be a reference configuration and χ be an arbitrary configuration of $\mathcal{B}$. Then the mapping

$$\chi_\kappa = \chi \circ \kappa^{-1} : \mathcal{B}_\kappa \to \mathcal{B}_\chi, \qquad \boldsymbol{x} = \chi_\kappa(\boldsymbol{X}) = \chi(\kappa^{-1}(\boldsymbol{X})), \tag{1.2}$$

is called the *deformation* of $\mathcal{B}$ from κ to χ (Fig. 1.1). In terms of coordinate systems $(x^i, i = 1, 2, 3)$ and $(X^\alpha, \alpha = 1, 2, 3)$ in the deformed and the reference configurations respectively, the deformation χ_κ is given by

$$x^i = \chi^i(X^\alpha), \tag{1.3}$$

where χ^i are called the *deformation functions*.

The *deformation gradient* of χ relative to κ, denoted by F_κ is defined by

$$F_\kappa = \nabla_{\boldsymbol{X}} \chi_\kappa. \tag{1.4}$$

Here we have assumed that χ_κ is differentiable. Furthermore, we shall require that F_κ is non-singular, so that by the inverse function theorem, the inverse mapping χ_κ^{-1} exists and is also differentiable. Therefore the determinant of F_κ must be different from zero,

$$J = \det F_\kappa \neq 0. \tag{1.5}$$

When the reference configuration κ is chosen and understood in the context, F_κ will be denoted simply by F.

Relative to the natural bases $\boldsymbol{e}^\alpha(\boldsymbol{X})$ and $\boldsymbol{e}_i(\boldsymbol{x})$ of the coordinate systems (X^α) and (x^i) respectively, the deformation gradient F can be expressed in the following component form,

$$F = F^i{}_\alpha \boldsymbol{e}_i(\boldsymbol{x}) \otimes \boldsymbol{e}^\alpha(\boldsymbol{X}), \qquad F^i{}_\alpha = \frac{\partial \chi^i}{\partial X^\alpha}. \tag{1.6}$$

Let $d\boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}_0$ be a small (infinitesimal) material line element in the reference configuration, and $d\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}) - \chi_\kappa(\boldsymbol{X}_0)$ be its image in the deformed configuration, then it follows from the definition that

$$d\boldsymbol{x} = F d\boldsymbol{X}, \tag{1.7}$$

since $d\boldsymbol{X}$ is infinitesimal the higher order term $o(d\boldsymbol{X})$ tends to zero.

Similarly, let da_κ and $\boldsymbol{n}_\kappa$ be a small material surface element and its unit normal in the reference configuration and da and $\boldsymbol{n}$ be the corresponding ones in the deformed configuration. And let dv_κ and dv be small material volume elements in the reference and the deformed configurations respectively. Then we have

$$\boldsymbol{n}\, da = JF^{-T}\boldsymbol{n}_\kappa da_\kappa, \qquad dv = |J|\; dv_\kappa. \tag{1.8}$$

1.2 Strain and rotation

The deformation gradient is a measure of local deformation of the body. We shall introduce other measures of deformation which have more suggestive physical meanings, such as change of shape and orientation. First we shall recall the following theorem from linear algebra:

Theorem (polar decomposition)**.** *For any non-singular tensor F, there exist unique symmetric positive definite tensors V and U and a unique orthogonal tensor R such that*

$$F = RU = VR. \tag{1.9}$$

Since the deformation gradient F is non-singular, the above decomposition holds. We observe that a positive definite symmetric tensor represents a state of pure stretches along three mutually orthogonal axes and an orthogonal tensor a rotation. Therefore, (1.9) assures that any local deformation is a combination of a pure stretch and a rotation.

We call R the *rotation tensor*, while U and V are called the *right* and the *left stretch tensors* respectively. Both stretch tensors measure the local strain, a change of shape, while the tensor R describes the local rotation, a change of orientation, experienced by material elements of the body.

Clearly we have

$$\begin{aligned} &U^2 = F^TF, \qquad V^2 = FF^T, \\ &\det U = \det V = |\det F|. \end{aligned} \tag{1.10}$$

Since $V = RUR^T$, V and U have the same eigenvalues and their eigenvectors differ only by the rotation R. Their eigenvalues are called the *principal stretches,* and the corresponding eigenvectors are called the *principal directions.*

We shall also introduce the *right* and the *left Cauchy-Green strain tensors* defined by

$$C = F^TF, \qquad B = FF^T, \tag{1.11}$$

respectively, which are easier to be calculated than the strain measures U and V from a given F in practice. Note that C and U share the same eigenvectors, while the eigenvalues of U are the positive square root of those of C; the same is true for B and V.

1.3 Linear strain tensors

The strain tensors introduced in the previous section are valid for finite deformations in general. In the classical linear theory, only small deformations are considered.

We introduce the *displacement vector* from the reference configuration (see Fig. 1.2),

$$\boldsymbol{u} = \chi_\kappa(\boldsymbol{X}) - \boldsymbol{X},$$

and its gradient,

$$H = \nabla_{\boldsymbol{X}} \boldsymbol{u}.$$

Obviously, we have $F = 1 + H$.

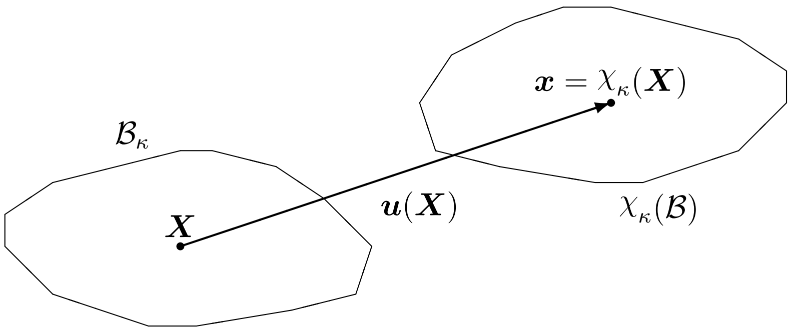

Figure 1.2: Displacement vector

For small deformations, the displacement gradient H is assumed to be a small quantity, $|H| \ll 1$, and say H is of order $o(1)$. The right stretch tensor U and the rotation tensor R can then be approximated by

$$\begin{aligned} U &= \sqrt{F^T F} = 1 + \tfrac{1}{2}(H + H^T) + o(2) = 1 + E + o(2), \\ R &= F U^{-1} = 1 + \tfrac{1}{2}(H - H^T) + o(2) = 1 + \widetilde{R} + o(2), \end{aligned} \tag{1.12}$$

where

$$E = \frac{1}{2}(H + H^T), \qquad \widetilde{R} = \frac{1}{2}(H - H^T), \tag{1.13}$$

are called the *infinitesimal strain tensor* and the *infinitesimal rotation tensor*, respectively. Note that infinitesimal strain and rotation are the symmetric and skew-symmetric parts of the displacement gradient.

We can give geometrical meanings to the components of the infinitesimal strain tensor E_{ij} relative to a Cartesian coordinate system. Consider two infinitesimal material line segments $d\boldsymbol{X}_1$ and $d\boldsymbol{X}_2$ in the reference configuration and their corresponding ones $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ in the current configuration. By (1.7), we have

$$d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 - d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2 = (F^T F - 1) d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2 = 2E\, d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2, \tag{1.14}$$

for small deformations. Now let $d\boldsymbol{X}_1 = d\boldsymbol{X}_2 = s_o\boldsymbol{e}_1$ be a small material line segment in the direction of the unit base vector $\boldsymbol{e}_1$ and s be the deformed length. Then we have

$$s^2 - s_o^2 = 2s_o^2\,(E\boldsymbol{e}_1 \cdot \boldsymbol{e}_1) = 2s_o^2\,E_{11},$$

which implies that

$$E_{11} = \frac{s^2 - s_o^2}{2s_o^2} = \frac{(s-s_o)(s+s_o)}{2s_o^2} \simeq \frac{s - s_o}{s_o}.$$

In other words, E_{11} is the change of length per unit original length of a small line segment in the $\boldsymbol{e}_1$-direction. The other diagonal components, E_{22} and E_{33} have similar interpretations as elongation per unit original length in their respective directions.

Similarly, let $d\boldsymbol{X}_1 = s_o\boldsymbol{e}_1$ and $d\boldsymbol{X}_2 = s_o\boldsymbol{e}_2$ and denote the angle between the two line segments after deformation by θ. Then we have

$$s_o^2\,|F\boldsymbol{e}_1|\,|F\boldsymbol{e}_2|\,\cos\theta - s_o^2\cos\frac{\pi}{2} = 2s_o^2\,(E\boldsymbol{e}_1 \cdot \boldsymbol{e}_2),$$

from which, if we write $\gamma = \pi/2 - \theta$, the change from its original right angle, then

$$\frac{\sin\gamma}{2} = \frac{E_{12}}{|F\boldsymbol{e}_1|\,|F\boldsymbol{e}_2|}.$$

Since $|E_{12}| \ll 1$ and $|F\boldsymbol{e}_i| \simeq 1$, it follows that $\sin\gamma \simeq \gamma$ and we conclude that

$$E_{12} \simeq \frac{\gamma}{2}.$$

Therefore, the component E_{12} is equal to one-half the change of angle between the two line segments originally along the $\boldsymbol{e}_1$- and $\boldsymbol{e}_2$-directions. Other off-diagonal components, E_{23} and E_{13} have similar interpretations as change of angle indicated by their numerical subscripts.

Moreover, since $\det F = \det(1 + H) \simeq 1 + \operatorname{tr} H$ for small deformations, by $(1.8)_2$ for a small material volume we have

$$\operatorname{tr} E = \operatorname{tr} H \simeq \frac{dv - dv_\kappa}{dv_\kappa}.$$

Thus the sum $E_{11} + E_{22} + E_{33}$ measures the infinitesimal change of volume per unit original volume. Therefore, in the linear theory, if the deformation is incompressible, since $\operatorname{tr} E = \operatorname{Div} \boldsymbol{u}$, it follows that

$$\operatorname{Div} \boldsymbol{u} = 0. \tag{1.15}$$

In terms of Cartesian coordinates, the displacement gradient

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \frac{\partial u_i}{\partial x_k}(\delta_{jk} + o(1)) = \frac{\partial u_i}{\partial x_j} + o(2),$$

in other words, the two displacement gradients

$$\frac{\partial u_i}{\partial X_j} \quad \text{and} \quad \frac{\partial u_i}{\partial x_j}$$

differ in second order terms only. Therefore, since in the classical linear theory, the higher order terms are insignificant, it is usually not necessary to introduce the reference configuration in the linear theory. The classical infinitesimal strain and rotation, in the Cartesian coordinate system, are defined as

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \qquad \widetilde{R}_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}), \tag{1.16}$$

in the current configuration.

1.4 Motions

A *motion* of the body $\mathcal{B}$ can be regarded as a continuous sequence of deformations in time, *i.e.*, a *motion* χ of $\mathcal{B}$ is regarded as a map,

$$\chi : \mathcal{B}_\kappa \times I\!R \to \mathcal{E}, \qquad \boldsymbol{x} = \chi(\boldsymbol{X}, t). \tag{1.17}$$

We denote the configuration of $\mathcal{B}$ at time t in the motion χ by $\mathcal{B}_t$.

In practice, the reference configuration κ is often chosen as the configuration in the motion at some instant t_0, $\kappa = \chi(\cdot, t_0)$, say for example, $t_0 = 0$, so that $\boldsymbol{X} = \chi(\boldsymbol{X}, 0)$. But such a choice is not necessary in general. The reference configuration κ can be chosen independently of any motion.

For a fixed material point $\boldsymbol{X}$,

$$\chi(\boldsymbol{X}, \cdot\,) : I\!R \to \mathcal{E}$$

is a curve called the *path* of the material point $\boldsymbol{X}$. The *velocity* $\boldsymbol{v}$ and the *acceleration* $\boldsymbol{a}$ are defined as the first and the second time derivatives of position as $\boldsymbol{X}$ moves along its path,

$$\boldsymbol{v} = \frac{\partial \chi(\boldsymbol{X}, t)}{\partial t}, \qquad \boldsymbol{a} = \frac{\partial^2 \chi(\boldsymbol{X}, t)}{\partial t^2}. \tag{1.18}$$

The velocity and the acceleration are vector quantities. Here of course, we have assumed that $\chi(\boldsymbol{X}, t)$ is twice differentiable with respect to t. However, from now on, we shall assume that all functions are smooth enough for the conditions needed in the context, without their smoothness explicitly specified.

A material body is endowed with some physical properties whose values may change along with the deformation of the body in a motion. A quantity defined on a motion can be described in essentially two different ways: either by the evolution of its value along the path of a material point or by the change of its value at a fixed location in the deformed body. The former is called the material (or a referential description if a reference configuration is used) and the later a spatial description. We shall make them more precise below.

For a given motion χ and a fixed reference configuration κ, consider a quantity, with its value in some space W, defined on the motion of $\mathcal{B}$ by a function

$$f : \mathcal{B} \times I\!R \to W. \tag{1.19}$$

Then it can be defined on the reference configuration,

$$\hat{f} : \mathcal{B}_\kappa \times I\!R \to W, \tag{1.20}$$

by

$$\hat{f}(\boldsymbol{X}, t) = f(\kappa^{-1}(\boldsymbol{X}), t) = f(\boldsymbol{p}, t), \qquad \boldsymbol{X} \in \mathcal{B}_\kappa,$$

and also defined on the position occupied by the body at time t,

$$\tilde{f}(\cdot, t) : \mathcal{B}_t \to W, \tag{1.21}$$

by

$$\tilde{f}(\boldsymbol{x}, t) = \hat{f}(\chi^{-1}(\boldsymbol{x}, t), t) = \hat{f}(\boldsymbol{X}, t), \qquad \boldsymbol{x} \in \mathcal{B}_t.$$

As a custom in continuum mechanics, one usually denotes these functions f, $\hat{f}$, and $\tilde{f}$ by the same symbol since they have the same value at the corresponding point, and write, by an abuse of notations,

$$f = f(\boldsymbol{p}, t) = f(\boldsymbol{X}, t) = f(\boldsymbol{x}, t),$$

and called them respectively the *material description*, the *referential description* and the *spatial description* of the function f. Sometimes the referential description is referred to as the *Lagrangian description* and the spatial description as the *Eulerian description*.

When a reference configuration is chosen and fixed, one can usually identify the material point $\boldsymbol{p}$ with its reference position $\boldsymbol{X}$. In fact, the material description in $(\boldsymbol{p}, t)$ is rarely used and the referential description in $(\boldsymbol{X}, t)$ is often regarded as *the* material description instead. However, in later discussions concerning material frame-indifference of constitutive equations, we shall emphasize the difference between the material description and a referential description, because the true nature of material properties should not depend on the choice of a reference configuration.

Possible confusions may arise in such an abuse of notations, especially when differentiations are involved. To avoid such confusions, one may use different notations for differentiation in these situations.

In the referential description, the time derivative is denoted by a dot while the differential operators such as gradient, divergence and curl are denoted by Grad, Div and Curl respectively, beginning with capital letters:

$$\dot{f} = \frac{\partial f(\boldsymbol{X}, t)}{\partial t}, \qquad \operatorname{Grad} f = \nabla_{\boldsymbol{X}} f(\boldsymbol{X}, t), \text{ etc.}$$

In the spatial description, the time derivative is the usual ∂_t and the differential operators beginning with lower-case letters, grad, div and curl:

$$\frac{\partial f}{\partial t} = \frac{\partial f(\boldsymbol{x}, t)}{\partial t}, \qquad \operatorname{grad} f = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}, t), \text{ etc.}$$

The relations between these notations can easily be obtained. Indeed, let f be a scalar field and $\boldsymbol{u}$ be a vector field. We have

$$\dot{f} = \frac{\partial f}{\partial t} + (\operatorname{grad} f) \cdot \boldsymbol{v}, \qquad \dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t} + (\operatorname{grad} \boldsymbol{u})\boldsymbol{v}, \tag{1.22}$$

and

$$\operatorname{Grad} f = F^T \operatorname{grad} f, \qquad \operatorname{Grad} \boldsymbol{u} = (\operatorname{grad} \boldsymbol{u})F. \tag{1.23}$$

In particular, taking the velocity $\boldsymbol{v}$ for $\boldsymbol{u}$, it follows that

$$\operatorname{grad} \boldsymbol{v} = \dot{F}F^{-1}, \tag{1.24}$$

since $\operatorname{Grad} \boldsymbol{v} = \operatorname{Grad} \dot{\boldsymbol{x}} = \dot{F}$.

We call $\dot{f}$ the *material time derivative* of f, which is the time derivative of f following the path of the material point. Therefore, by the definition (1.18), we can write the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$ as

$$\boldsymbol{v} = \dot{\boldsymbol{x}}, \qquad \boldsymbol{a} = \ddot{\boldsymbol{x}},$$

and hence by $(1.22)_2$,

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial t} + (\operatorname{grad} \boldsymbol{v})\boldsymbol{v}. \tag{1.25}$$

1.5 Rate of deformation

Whereas the deformation gradient measures the local deformation, the material time derivative of deformation gradient measures the rate at which such changes occur at a fixed material point. Another measure for the rate of deformation relative to the current configuration at a fixed location, more commonly used in fluid mechanics, will be introduced here. To do it, we choose the current configuration $\chi(\cdot, t)$ as the reference configuration so that past and future deformations can be described relative to the present configuration.

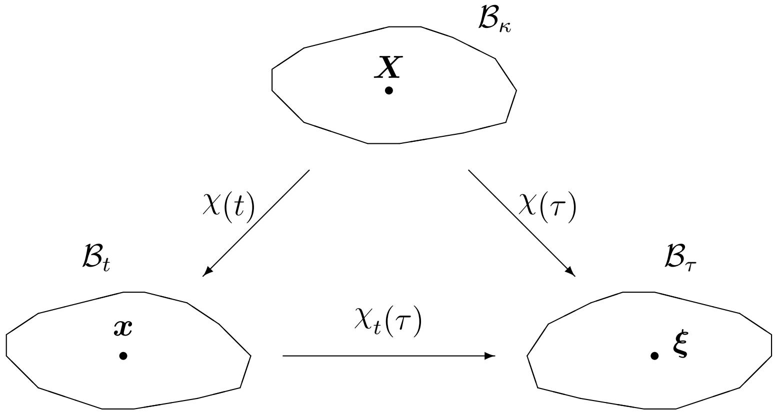

Figure 1.3: Relative deformation

We denote the position of the material point $\boldsymbol{X} \in \mathcal{B}_\kappa$ at time τ by $\boldsymbol{\xi}$,

$$\boldsymbol{\xi} = \chi(\boldsymbol{X}, \tau).$$

Then

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t), \qquad \boldsymbol{\xi} = \chi_t(\boldsymbol{x}, \tau) = \chi(\chi^{-1}(\boldsymbol{x}, t), \tau), \tag{1.26}$$

where $\chi_t(\cdot,\tau) : \mathcal{B}_t \to \mathcal{B}_\tau$ is the deformation at time τ relative to the configuration at time t or simply called the *relative deformation* (Fig. 1.3). The *relative deformation gradient* F_t is defined by

$$F_t(\boldsymbol{x},\tau) = \nabla_{\boldsymbol{x}}\chi_t(\boldsymbol{x},\tau), \tag{1.27}$$

that is, the deformation gradient at time τ with respect to the configuration at time t. Of course, if $\tau = t$,

$$F_t(\boldsymbol{x},t) = 1,$$

and we can easily show that

$$F(\boldsymbol{X},\tau) = F_t(\boldsymbol{x},\tau)F(\boldsymbol{X},t). \tag{1.28}$$

The rate of change of deformation relative to the current configuration is defined as

$$L(\boldsymbol{x},t) = \frac{\partial}{\partial\tau}F_t(\boldsymbol{x},\tau)\Big|_{\tau=t}.$$

From (1.28), $F_t(\tau) = F(\tau)F(t)^{-1}$, by taking the derivative with respect to τ, we have

$$\begin{aligned}\dot{F}_t(\tau) &= \dot{F}(\tau)F(t)^{-1}\\ &= (\operatorname{grad}\boldsymbol{v}(\tau))F(\tau)F(t)^{-1} = (\operatorname{grad}\boldsymbol{v}(\tau))F_t(\tau),\end{aligned}$$

and since $F_t(t) = 1$, we conclude that

$$L = \operatorname{grad}\boldsymbol{v}. \tag{1.29}$$

In other words, the spatial velocity gradient can also be interpreted as the rate of change of deformation relative to the current configuration.

Moreover, if we apply the polar decomposition to the relative deformation gradient $F_t(\boldsymbol{x},\tau)$,

$$F_t = R_tU_t = V_tR_t, \tag{1.30}$$

by holding $\boldsymbol{x}$ and t fixed and taking the derivative of F_t with respect to τ, we obtain

$$\dot{F}_t(\tau) = R_t(\tau)\dot{U}_t(\tau) + \dot{R}_t(\tau)U_t(\tau),$$

and hence by putting $\tau = t$, we have

$$L(t) = \dot{U}_t(t) + \dot{R}_t(t). \tag{1.31}$$

If we denote

$$D(t) = \dot{U}_t(t), \qquad W(t) = \dot{R}_t(t), \tag{1.32}$$

We can show easily that

$$D^T = D, \qquad W^T = -W. \tag{1.33}$$

Therefore, the relation (1.31) is just a decomposition of the tensor L into its symmetric and skew-symmetric parts, and by (1.29) we have

$$\begin{aligned}D &= \frac{1}{2}(\operatorname{grad}\boldsymbol{v} + \operatorname{grad}\boldsymbol{v}^T),\\ W &= \frac{1}{2}(\operatorname{grad}\boldsymbol{v} - \operatorname{grad}\boldsymbol{v}^T).\end{aligned} \tag{1.34}$$

In view of (1.32) the symmetric part of the velocity gradient, D, is called the *rate of strain tensor* or simply as the *stretching tensor*, and the skew-symmetric part of the velocity gradient, W, is called the *rate of rotation tensor* or simply as the *spin tensor*.

Since the spin tensor W is skew-symmetric, it can be represented as an axial vector $\boldsymbol{w}$. The components of the vector $\boldsymbol{w}$ are usually defined by $w^i = \varepsilon_{ijk} W_{kj}$ in the Cartesian coordinate system, hence it follow that

$$\boldsymbol{w} = \operatorname{curl} \boldsymbol{v}. \tag{1.35}$$

The vector $\boldsymbol{w}$ is called the *vorticity* vector in fluid dynamics.

CHAPTER 2

Balance Laws

2.1 General balance equation

Basic laws of mechanics can all be expressed in general in the following form,

$$\frac{d}{dt}\int_{\mathcal{P}_t} \psi \, dv = \int_{\partial\mathcal{P}_t} \mathit{\Phi}_\psi \boldsymbol{n} \, da + \int_{\mathcal{P}_t} \sigma_\psi \, dv, \tag{2.1}$$

for any bounded regular subregion of the body, called a part $\mathcal{P} \subset \mathcal{B}$ and the vector field $\boldsymbol{n}$, the outward unit normal to the boundary of the region $\mathcal{P}_t$ in the current configuration. The quantities ψ and σ_ψ are tensor fields of certain order m, and $\mathit{\Phi}_\psi$ is a tensor field of order $m+1$, say $m = 0$ or $m = 1$ so that ψ is a scalar or vector quantity, and respectively $\mathit{\Phi}_\psi$ is a vector or second order tensor quantity.

The relation (2.1), called the *general balance* of ψ in integral form, is interpreted as asserting that the rate of increase of the quantity ψ in a part $\mathcal{P}$ of a body is affected by the inflow of ψ through the boundary of $\mathcal{P}$ and the growth of ψ within $\mathcal{P}$. We call $\mathit{\Phi}_\psi$ the *flux* of ψ and σ_ψ the *supply* of ψ.

We are interested in the local forms of the integral balance (2.1) at a point in the region $\mathcal{P}_t$. The derivation of local forms rest upon certain assumptions on the smoothness of the tensor fields ψ, $\mathit{\Phi}_\psi$, and σ_ψ. Here not only regular points, where all the tensor fields are smooth, but also singular points, where they may suffer jump discontinuities, will be considered.

First of all, we need the following theorem, which is a three-dimensional version of the formula in calculus for differentiation under the integral sign on a moving interval, namely

$$\frac{\partial}{\partial t}\int_{g(t)}^{f(t)} \psi(x,t)\, dx = \int_{g(t)}^{f(t)} \frac{\partial \psi}{\partial t}\, dx + \psi(f(t),t)\, \dot{f}(t) - \psi(g(t),t)\, \dot{g}(t).$$

Theorem (transport theorem)**.** *Let $V(t)$ be a regular region in $\mathcal{E}$ and $u_n(\boldsymbol{x},t)$ be the outward normal speed of a surface point $\boldsymbol{x} \in \partial V(t)$. Then for any smooth tensor field $\psi(\boldsymbol{x},t)$, we have*

$$\frac{d}{dt}\int_V \psi \, dv = \int_V \frac{\partial \psi}{\partial t}\, dv + \int_{\partial V} \psi \, u_n \, da. \tag{2.2}$$

In this theorem, the surface speed $u_n(\boldsymbol{x},t)$ needs only to be defined on the boundary ∂V. If $V(t)$ is a material region $\mathcal{P}_t$, *i.e.*, it always consists of the same material points of a part $\mathcal{P} \subset \mathcal{B}$, then $u_n = \dot{\boldsymbol{x}} \cdot \boldsymbol{n}$ and (2.2) becomes

$$\frac{d}{dt} \int_{\mathcal{P}_t} \psi \, dv = \int_{\mathcal{P}_t} \frac{\partial \psi}{\partial t} \, dv + \int_{\partial \mathcal{P}_t} \psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} \, da. \tag{2.3}$$

Now we shall extend the above transport theorem to a material region containing a surface across which ψ may suffer a jump discontinuity.

Definition. *An oriented smooth surface $\mathcal{S}$ in a material region $\mathcal{V}$ is called a* singular surface *relative to a field ϕ defined on $\mathcal{V}$, if ϕ is smooth in $\mathcal{V} - \mathcal{S}$ and suffers a jump discontinuity across $\mathcal{S}$. The* jump *of ϕ is defined as*

$$[\![\phi]\!] = \phi^+ - \phi^-, \tag{2.4}$$

where ϕ^+ and ϕ^- are the one side limits from the two regions of $\mathcal{V}$ separated by $\mathcal{S}$ and designated as $\mathcal{V}^+$ and $\mathcal{V}^-$ respectively.

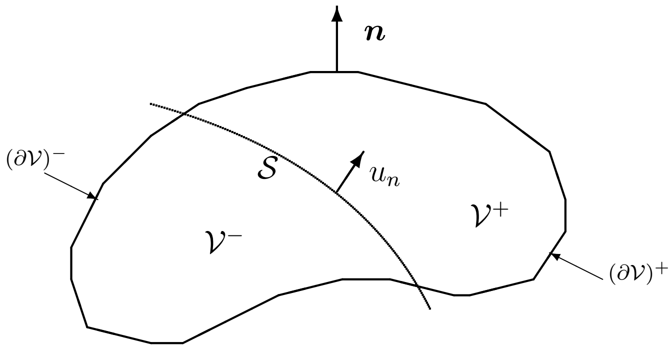

Figure 2.1: Singular surface

Let u_n be the normal speed of $\mathcal{S}$ with the direction pointing into $\mathcal{V}^+$ and $\boldsymbol{n}$ be the outward unit normal of $\partial\mathcal{V}$ (Fig. 2.1). Denote

$$(\partial\mathcal{V})^\pm = \partial\mathcal{V}^\pm \cap \partial\mathcal{V}.$$

Since both $(\partial\mathcal{V})^+$ and $(\partial\mathcal{V})^-$ are material surfaces, their normal surface speed is $\dot{\boldsymbol{x}} \cdot \boldsymbol{n}$. Clearly,

$$\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-, \quad \partial\mathcal{V} = (\partial\mathcal{V})^+ \cup (\partial\mathcal{V})^-. \tag{2.5}$$

Since $\mathcal{S}$ need not be a material surface, $\mathcal{V}^+$ and $\mathcal{V}^-$ need not be material regions in general.

Suppose that $\psi(\boldsymbol{x}, t)$ and $\dot{\boldsymbol{x}}(\boldsymbol{x}, t)$ are smooth in $\mathcal{V}^+$ and $\mathcal{V}^-$, then the transport theorem (2.2) implies that

$$\frac{d}{dt} \int_{\mathcal{V}^+} \psi \, dv = \int_{\mathcal{V}^+} \frac{\partial \psi}{\partial t} \, dv + \int_{(\partial\mathcal{V})^+} \psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} \, da + \int_{\mathcal{S}} \psi^+ (-u_n) \, da, \tag{2.6}$$

$$\frac{d}{dt} \int_{\mathcal{V}^-} \psi \, dv = \int_{\mathcal{V}^-} \frac{\partial \psi}{\partial t} \, dv + \int_{(\partial\mathcal{V})^-} \psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} \, da + \int_{\mathcal{S}} \psi^- u_n \, da. \tag{2.7}$$

Adding (2.6) and (2.7), we obtain by the use of (2.5) the following transport theorem in a material region containing a singular surface:

Theorem. *Let $V(t)$ be a material region in $\mathcal{E}$ and $\mathcal{S}(t)$ be a singular surface relative to the tensor field $\psi(\boldsymbol{x}, t)$ which is smooth elsewhere. Then we have*

$$\frac{d}{dt}\int_{\mathcal{V}} \psi \, dv = \int_{\mathcal{V}} \frac{\partial \psi}{\partial t} \, dv + \int_{\partial \mathcal{V}} \psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} \, da - \int_{\mathcal{S}} [\![\psi]\!] \, u_n \, da, \tag{2.8}$$

where $u_n(\boldsymbol{x}, t)$ is the normal speed of a surface point $\boldsymbol{x} \in \mathcal{S}(t)$ and $[\![\psi]\!]$ is the jump of ψ across $\mathcal{S}$.

2.2 Local balance equation and jump condition

For a material region $\mathcal{V}$ containing a singular surface $\mathcal{S}$, the equation of general balance in integral form (2.1) becomes

$$\int_{\mathcal{V}} \frac{\partial \psi}{\partial t} \, dv + \int_{\partial \mathcal{V}} \psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} \, da - \int_{\mathcal{S}} [\![\psi]\!] \, u_n \, da = \int_{\partial \mathcal{V}} \Phi_\psi \boldsymbol{n} \, da + \int_{\mathcal{V}} \sigma_\psi \, dv. \tag{2.9}$$

Definition. *A point $\boldsymbol{x}$ is called* regular *if there is a material region containing $\boldsymbol{x}$ in which all the tensor fields in (2.1) are smooth. And a point $\boldsymbol{x}$ is called* singular *if it is a point on a singular surface relative to ψ and Φ_ψ.*

We can obtain the local balance equation at a regular point as well as at a singular point from the above integral equation. First, we consider a small material region $\mathcal{V}$ containing $\boldsymbol{x}$ such that $\mathcal{V} \cap \mathcal{S} = \emptyset$. By the use of the divergence theorem, (2.9) becomes

$$\int_{\mathcal{V}} \Big\{ \frac{\partial \psi}{\partial t} + \operatorname{div}(\psi \otimes \dot{\boldsymbol{x}} - \Phi_\psi) - \sigma_\psi \Big\} \, dv = 0. \tag{2.10}$$

Since the integrand is smooth and the equation (2.10) holds for any $\mathcal{V}$, such that $\boldsymbol{x} \in \mathcal{V}$, and $\mathcal{V} \cap \mathcal{S} = \emptyset$, the integrand must vanish at $\boldsymbol{x}$. Therefore we have

Theorem (local balance equation)**.** *At a regular point $\boldsymbol{x}$, the general balance equation (2.9) reduces to*

$$\frac{\partial \psi}{\partial t} + \operatorname{div}(\psi \otimes \dot{\boldsymbol{x}} - \Phi_\psi) - \sigma_\psi = 0. \tag{2.11}$$

The notation $\operatorname{div}(\psi \otimes \dot{\boldsymbol{x}})$ in (2.11) should be understood as $\operatorname{div}(\psi \, \dot{\boldsymbol{x}})$ when ψ is a scalar quantity.

Next, we consider a singular point $\boldsymbol{x}$, *i.e.,* $\boldsymbol{x} \in \mathcal{S}$. Let $\mathcal{V}$ be an arbitrary material region around $\boldsymbol{x}$, and $s = \mathcal{V} \cap \mathcal{S}$ (Fig. 2.2). We shall take the limit by shrinking $(\partial \mathcal{V})^+$ and $(\partial \mathcal{V})^-$ down to s in such a way that the volume of $\mathcal{V}$ tends to zero, while the area of s remain unchanged. If $\partial_t \psi$ and σ_ψ are bounded in $\mathcal{V}$ then the volume integrals vanish in the limit and (2.9) becomes

$$\int_s \{ [\![\psi \, \dot{\boldsymbol{x}} \cdot \boldsymbol{n} - \Phi_\psi \boldsymbol{n}]\!] - [\![\psi]\!] \, u_n \} \, da = 0. \tag{2.12}$$

Since the integrand is smooth on s and (2.12) holds for any s containing $\boldsymbol{x}$, the integrand must vanish at $\boldsymbol{x}$. We obtain

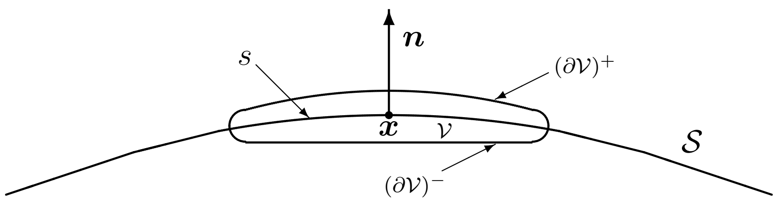

Figure 2.2: At a singular point

Theorem (jump condition)**.** *At a singular point* $\boldsymbol{x}$*, the general balance equation* (2.9) *reduces to*

$$[\![\psi\,\dot{\boldsymbol{x}}\cdot\boldsymbol{n}-\Phi_\psi\boldsymbol{n}]\!]-[\![\psi]\!]\,u_n=0, \tag{2.13}$$

if in addition, $\dfrac{\partial\psi}{\partial t}$ *and* σ_ψ *are bounded in the neighborhood of* $\boldsymbol{x}$*.*

The jump condition (2.13) is also known as the Rankine-Hugoniot equation. It can also be written as

$$[\![\psi\,U]\!]+[\![\Phi_\psi]\!]\,\boldsymbol{n}=0, \tag{2.14}$$

where

$$U^\pm=u_n-\dot{\boldsymbol{x}}^\pm\cdot\boldsymbol{n} \tag{2.15}$$

are called the *local speeds of propagation* of $\mathcal{S}$ relative to the motion of the body. If $\mathcal{S}$ is a material surface then $U^\pm=0$.

2.3 Balance equations in reference coordinates

Sometimes, for solid bodies, it is more convenient to use the referential description. The corresponding relations for the balance equation (2.11) and the jump condition (2.13) in the reference coordinates can be derived in a similar manner. We begin with the integral form (2.1) now written in the reference configuration κ,

$$\frac{d}{dt}\int_{\mathcal{P}_\kappa}\psi_\kappa\,dv_\kappa=\int_{\partial\mathcal{P}_\kappa}\Phi^\psi_\kappa\boldsymbol{n}_\kappa\,da_\kappa+\int_{\mathcal{P}_\kappa}\sigma^\psi_\kappa\,dv_\kappa. \tag{2.16}$$

In view of the relations for volume elements and surface elements (1.8), the corresponding quantities are defined as

$$\psi_\kappa=|J|\,\psi,\quad \Phi^\psi_\kappa=J\Phi_\psi F^{-T},\quad \sigma^\psi_\kappa=|J|\,\sigma_\psi. \tag{2.17}$$

The transport theorem (2.2) remains valid for $\psi_\kappa(\boldsymbol{X},t)$ in a movable region $V(t)$,

$$\frac{d}{dt}\int_V\psi_\kappa dv_\kappa=\int_V\dot{\psi}_\kappa dv_\kappa+\int_{\partial V}\psi_\kappa U_\kappa da_\kappa, \tag{2.18}$$

where $U_\kappa(\boldsymbol{X},t)$ is the outward normal speed of a surface point $\boldsymbol{X} \in \partial V(t)$. Therefore for singular surface $\mathcal{S}_\kappa(t)$ moving across a material region $\mathcal{V}_\kappa$ in the reference configuration, by a similar argument, we can obtain from (2.9)

$$\int_{\mathcal{V}_\kappa} \dot{\psi}_\kappa dv_\kappa - \int_{\mathcal{S}_\kappa} [\![\psi_\kappa]\!]\, U_\kappa da_\kappa = \int_{\partial\mathcal{V}_\kappa} \Phi^\psi_\kappa \boldsymbol{n}_\kappa da_\kappa + \int_{\mathcal{V}_\kappa} \sigma^\psi_\kappa dv_\kappa, \tag{2.19}$$

where U_κ is the normal speed of the surface points on $\mathcal{S}_\kappa$, while the normal speed of the surface points on $\partial\mathcal{V}_\kappa$ is zero since a material region is a fixed region in the reference configuration. From this equation, we obtain the local balance equation and the jump condition in the reference configuration,

$$\dot{\psi}_\kappa - \operatorname{Div} \Phi^\psi_\kappa - \sigma^\psi_\kappa = 0, \qquad [\![\psi_\kappa]\!]\, U_\kappa + [\![\Phi^\psi_\kappa]\!]\, \boldsymbol{n}_\kappa = 0, \tag{2.20}$$

where U_κ and $\boldsymbol{n}_\kappa$ are the normal speed and the unit normal vector of the singular surface.

2.4 Conservation of mass

Let $\rho(\boldsymbol{x},t)$ denote the mass density of $\mathcal{B}_t$ in the current configuration. Since the material is neither destroyed nor created in any motion in the absence of chemical reactions, we have

Conservation of mass. *The total mass of any part $\mathcal{P} \subset \mathcal{B}$ does not change in any motion,*

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\, dv = 0. \tag{2.21}$$

By comparison, it is a special case of the general balance equation (2.1) with no flux and no supply,

$$\psi = \rho, \qquad \Phi_\psi = 0, \qquad \sigma_\psi = 0,$$

and hence from (2.11) and (2.13) we obtain the local expressions for mass conservation and the jump condition at a singular point,

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\,\dot{\boldsymbol{x}}) = 0, \qquad [\![\rho\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{n} - u_n)]\!] = 0. \tag{2.22}$$

The equation (2.21) states that the total mass of any part is constant in time. In particular, if $\rho_\kappa(\boldsymbol{X})$ denote the mass density of $\mathcal{B}_\kappa$ in the reference configuration, than

$$\int_{\mathcal{P}_\kappa} \rho_\kappa dv_\kappa = \int_{\mathcal{P}_t} \rho\, dv, \tag{2.23}$$

which implies that

$$\rho_\kappa = \rho\,|J|, \tag{2.24}$$

where $J = \det F$. This is another form of the conservation of mass in the referential description, which also follows from the general expression (2.20) and (2.17).

We note that if the motion is volume-preserving, then

$$\frac{d}{dt}\int_{\mathcal{P}_t} dv = 0,$$

which by comparison with (2.1) again leads to

$$\operatorname{div} \dot{\boldsymbol{x}} = 0,$$

which in turn implies that $\dot{\rho} = 0$ by $(2.22)_1$. Therefore, for an incompressible motion, the mass density is time-independent and the divergence of velocity field must vanish.

2.5 Equation of motion

For a deformable body, the linear momentum and the angular momentum with respect to a point $\boldsymbol{x}_\circ \in \mathcal{E}$ of a part $\mathcal{P} \subset \mathcal{B}$ in the motion can be defined respectively as

$$\int_{\mathcal{P}_t} \rho\, \dot{\boldsymbol{x}}\, dv, \quad \text{and} \quad \int_{\mathcal{P}_t} \rho\, (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \dot{\boldsymbol{x}}\, dv.$$

In laying down the laws of motion, we follow the classical approach developed by Newton and Euler, according to which the change of momentum is produced by the action of forces. There are two type of forces, namely, one acts throughout the volume, called the *body force*, and one acts on the surface of the body, called the *surface traction*.

Euler's laws of motion. *Relative to an inertial frame, the motion of any part* $\mathcal{P} \subset \mathcal{B}$ *satisfies*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{P}_t} \rho\, \dot{\boldsymbol{x}}\, dv = \int_{\mathcal{P}_t} \rho\, \boldsymbol{b}\, dv + \int_{\partial\mathcal{P}_t} \boldsymbol{t}\, da,\\
&\frac{d}{dt}\int_{\mathcal{P}_t} \rho\, (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \dot{\boldsymbol{x}}\, dv = \int_{\mathcal{P}_t} \rho\, (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \boldsymbol{b}\, dv + \int_{\partial\mathcal{P}_t} (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \boldsymbol{t}\, da.
\end{aligned}$$

We remark that the existence of inertial frames (an equivalent of Newton's first law) is essential to establish the Euler's laws (equivalent of Newton's second law) in the above forms. Roughly speaking, a coordinate system $(\boldsymbol{x})$ at rest for $\mathcal{E}$ is usually regarded as an inertial frame. Euler's laws in an arbitrary frame can be obtained through the notion of change of reference frames considered in Chapter 3.

We call $\boldsymbol{b}$ the *body force density* (per unit mass), and $\boldsymbol{t}$ the *surface traction* (per unit surface area). Unlike the body force $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$, such as the gravitational force, the traction $\boldsymbol{t}$ at $\boldsymbol{x}$ depends, in general, upon the surface $\partial\mathcal{P}_t$ on which $\boldsymbol{x}$ lies. It is obvious that there are infinite many parts $\mathcal{P} \subset \mathcal{B}$, such that $\partial\mathcal{P}_t$ may also contain $\boldsymbol{x}$. However, following Cauchy, it is always assumed in the classical continuum mechanics that the tractions on all like-oriented surfaces with a common tangent plane at $\boldsymbol{x}$ are the same.

Postulate (Cauchy)**.** *Let* $\boldsymbol{n}$ *be the unit normal to the surface* $\partial\mathcal{P}_t$ *at* $\boldsymbol{x}$*, then*

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}). \tag{2.25}$$

An immediate consequence of this postulate is the well-known theorem which ensures the existence of stress tensor.

Theorem (Cauchy). *Suppose that* $\boldsymbol{t}(\cdot, \boldsymbol{n})$ *is a continuous function of* $\boldsymbol{x}$*, and* $\ddot{\boldsymbol{x}}$*,* $\boldsymbol{b}$ *are bounded in* $\mathcal{B}_t$*. Then Cauchy's postulate and Euler's first law implies the existence of a second order tensor* T*, such that*

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = T(\boldsymbol{x}, t)\,\boldsymbol{n}. \tag{2.26}$$

The tensor field $T(\boldsymbol{x}, t)$ in (2.26) is called the *Cauchy stress tensor*. The proof of the theorem can be found in most books of mechanics. The continuity assumption in the theorem is often too stringent for the existence of stress tensor in some applications, for example, problems involving shock waves. However, it has been shown that the theorem remains valid under a much weaker assumption of integrability, which would be satisfied in most applications [7, 6].

With (2.26) Euler's first law becomes

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\,\dot{\boldsymbol{x}}\,dv = \int_{\mathcal{P}_t} \rho\,\boldsymbol{b}\,dv + \int_{\partial\mathcal{P}_t} T\boldsymbol{n}\,da. \tag{2.27}$$

Comparison with the general balance equation (2.1) leads to

$$\psi = \rho\,\dot{\boldsymbol{x}}, \qquad \varPhi_\psi = T, \qquad \sigma_\psi = \rho\,\boldsymbol{b},$$

in this case, and hence from (2.11) and (2.13) we obtain the balance equation of linear momentum and its jump condition,

$$\frac{\partial}{\partial t}(\rho\,\dot{\boldsymbol{x}}) + \operatorname{div}(\rho\,\dot{\boldsymbol{x}} \otimes \dot{\boldsymbol{x}} - T) - \rho\,\boldsymbol{b} = 0, \qquad [\![\rho\,\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}} \cdot \boldsymbol{n} - u_n)]\!] - [\![T]\!]\boldsymbol{n} = 0. \tag{2.28}$$

The first equation, also known as the *equation of motion*, can be rewritten in the following more familiar form by the use of (2.22),

$$\rho\,\ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\,\boldsymbol{b}. \tag{2.29}$$

A similar argument for Euler's second law as a special case of (2.1) with

$$\psi = (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \rho\,\dot{\boldsymbol{x}}, \quad \varPhi_\psi\boldsymbol{n} = (\boldsymbol{x} - \boldsymbol{x}_\circ) \times T\boldsymbol{n}, \quad \sigma_\psi = (\boldsymbol{x} - \boldsymbol{x}_\circ) \times \rho\,\boldsymbol{b},$$

leads to

$$T = T^T, \tag{2.30}$$

after some simplification by the use of (2.29). In other words, the symmetry of the stress tensor is a consequence of the conservation of angular momentum.

2.6 Conservation of energy

Besides the kinetic energy, the total energy of a deformable body consists of another part called the internal energy,

$$\int_{\mathcal{P}_t} \Big(\rho\,\varepsilon + \frac{\rho}{2}\,\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}\Big)\,dv,$$

where $\varepsilon(\boldsymbol{x}, t)$ is called the specific *internal energy density*. The rate of change of the total energy is partly due to the mechanical power from the forces acting on the body and partly due to the energy inflow over the surface and the external energy supply.

Conservation of energy. *Relative to an inertial frame, the change of energy for any part $\mathcal{P} \subset \mathcal{B}$ is given by*

$$\frac{d}{dt}\int_{\mathcal{P}_t}(\rho\,\varepsilon + \frac{\rho}{2}\,\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})\,dv = \int_{\partial\mathcal{P}_t}(\dot{\boldsymbol{x}}\cdot T\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n})\,da + \int_{\mathcal{P}_t}(\rho\,\dot{\boldsymbol{x}}\cdot\boldsymbol{b} + \rho\,r)\,dv. \tag{2.31}$$

We call $\boldsymbol{q}(\boldsymbol{x},t)$ the *heat flux* vector (or energy flux), and $r(\boldsymbol{x},t)$ the *energy supply density* due to external sources, such as radiation. Comparison with the general balance equation (2.1), we have

$$\psi = (\rho\,\varepsilon + \frac{\rho}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}), \qquad \varPhi_\psi = T\,\dot{\boldsymbol{x}} - \boldsymbol{q}, \qquad \sigma_\psi = \rho\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{b} + r),$$

and hence we have the following local balance equation of total energy and its jump condition,

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\rho\,\varepsilon + \frac{\rho}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}\Big) + \operatorname{div}\Big((\rho\,\varepsilon + \frac{\rho}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})\dot{\boldsymbol{x}} + \boldsymbol{q} - T\,\dot{\boldsymbol{x}}\Big) = \rho\,(r + \dot{\boldsymbol{x}}\cdot\boldsymbol{b}),\\
&[\![(\rho\,\varepsilon + \frac{\rho}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})(\dot{\boldsymbol{x}}\cdot\boldsymbol{n} - u_n)]\!] + [\![\boldsymbol{q} - T\,\dot{\boldsymbol{x}}]\!]\cdot\boldsymbol{n} = 0.
\end{aligned} \tag{2.32}$$

The energy equation $(2.32)_1$ can be simplified by substracting the inner product of the equation of motion (2.29) with the velocity $\dot{\boldsymbol{x}}$,

$$\rho\,\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} = T\cdot\operatorname{grad}\dot{\boldsymbol{x}} + \rho\,r. \tag{2.33}$$

This is called the balance equation of internal energy. Note that the internal energy is not conserved and the term $T\cdot\operatorname{grad}\dot{\boldsymbol{x}}$ is the rate of work due to deformation.

Summary of basic equations:

By the use of material time derivative (1.22), the field equations can also be written as follows:

$$\begin{aligned}
&\dot{\rho} + \rho\,\operatorname{div}\boldsymbol{v} = 0,\\
&\rho\,\dot{\boldsymbol{v}} - \operatorname{div}T = \rho\,\boldsymbol{b},\\
&\rho\,\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} - T\cdot\operatorname{grad}\boldsymbol{v} = \rho\,r,
\end{aligned} \tag{2.34}$$

and the jump conditions in terms of local speed U of the singular surface become

$$\begin{aligned}
&[\![\rho\,U]\!] = 0,\\
&[\![\rho\,U\boldsymbol{v}]\!] + [\![T]\!]\boldsymbol{n} = 0,\\
&[\![\rho\,U(\varepsilon + \frac{1}{2}v^2)]\!] + [\![T\boldsymbol{v} - \boldsymbol{q}]\!]\cdot\boldsymbol{n} = 0,
\end{aligned} \tag{2.35}$$

where v^2 stands for $\boldsymbol{v}\cdot\boldsymbol{v}$.

2.7 Basic Equations in Material Coordinates

It is sometimes more convenient to rewrite the basic equations in material description relative to a reference configuration κ. They can easily be obtained from (2.20)$_1$,

$$\begin{aligned} &\rho = |J|^{-1}\rho_\kappa, \\ &\rho_\kappa \ddot{\boldsymbol{x}} = \operatorname{Div} T_\kappa + \rho_\kappa \boldsymbol{b}, \\ &\rho_\kappa \dot{\varepsilon} + \operatorname{Div} \boldsymbol{q}_\kappa = T_\kappa \cdot \dot{F} + \rho_\kappa r, \end{aligned} \tag{2.36}$$

where the following definitions have been introduced according to (2.17):

$$T_\kappa = J\,T F^{-T}, \qquad \boldsymbol{q}_\kappa = J F^{-1}\boldsymbol{q}. \tag{2.37}$$

T_κ is called the *Piola–Kirchhoff stress tensor* and $\boldsymbol{q}_\kappa$ is called the *material heat flux.* Note that unlike the Cauchy stress tensor T, the Piola–Kirchhoff stress tensor T_κ is not symmetric and it must satisfy

$$T_\kappa F^T = F\,T_\kappa^T. \tag{2.38}$$

The definition has been introduced according to the relation (1.8), which gives the relation,

$$\int_{\mathcal{S}} T\boldsymbol{n}\,da = \int_{\mathcal{S}_\kappa} T_\kappa \boldsymbol{n}_\kappa da_\kappa. \tag{2.39}$$

In other words, $T\boldsymbol{n}$ is the surface traction per unit area in the current configuration, while $T_\kappa \boldsymbol{n}_\kappa$ is the surface traction measured per unit area in the reference configuration. Similarly, $\boldsymbol{q}\cdot\boldsymbol{n}$ and $\boldsymbol{q}_\kappa\cdot\boldsymbol{n}_\kappa$ are the contact heat supplies per unit area in the current and the reference configurations respectively.

We can also write the jump conditions in the reference configuration by the use of (2.20)$_2$,

$$\begin{aligned} &[\![\rho_\kappa]\!]\,U_\kappa = 0, \\ &[\![\rho_\kappa \dot{\boldsymbol{x}}]\!]\,U_\kappa + [\![T_\kappa]\!]\,\boldsymbol{n}_\kappa = 0, \\ &[\![\rho_\kappa(\varepsilon + \tfrac{1}{2}\,\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})]\!]\,U_\kappa + [\![T_\kappa^T \dot{\boldsymbol{x}} - \boldsymbol{q}_\kappa]\!]\cdot\boldsymbol{n}_\kappa = 0, \end{aligned} \tag{2.40}$$

where $\boldsymbol{n}_\kappa$ and U_κ are the unit normal and the normal speed at the singular surface in the reference configuration.

2.8 Boundary conditions of a material body

The boundary of a material body is a material surface. Therefore, $U^\pm = 0$ at the boundary $\partial\mathcal{B}_t$, the jump conditions (2.35) of linear momentum and energy become

$$\begin{aligned} &[\![T]\!]\boldsymbol{n} = 0, \\ &[\![\boldsymbol{q}]\!]\cdot\boldsymbol{n} = [\![\boldsymbol{v}\cdot T\boldsymbol{n}]\!]. \end{aligned}$$

Suppose that the body is acted on the boundary by an external force $\boldsymbol{f}$ per unit area of the boundary $\partial\mathcal{B}_t$, then it requires the stress to satisfy the traction boundary condition,

$$T\boldsymbol{n} = \boldsymbol{f}, \tag{2.41}$$

and the heat flux to satisfy

$$[\![\boldsymbol{q}]\!] \cdot \boldsymbol{n} = [\![\boldsymbol{v}]\!] \cdot \boldsymbol{f}.$$

In particular, if either the boundary is fixed ($\boldsymbol{v} = 0$) or free ($\boldsymbol{f} = 0$) then the normal component of heat flux must be continuous at the boundary,

$$[\![\boldsymbol{q}]\!] \cdot \boldsymbol{n} = 0. \tag{2.42}$$

Therefore, if the body has a fixed adiabatic (meaning thermally isolated) boundary then the normal component of the heat flux must vanish at the boundary.

Since the deformed configuration is usually unknown for traction boundary value problems of a solid body, the above conditions, which involved the unknown boundary, are sometimes inconvenient.

Of course, we can also express the boundary conditions in the reference configuration. From the jump conditions $(2.40)_{2,3}$, we have

$$T_\kappa \boldsymbol{n}_\kappa = \boldsymbol{f}_\kappa, \qquad \boldsymbol{q}_\kappa \cdot \boldsymbol{n}_\kappa = h_\kappa, \tag{2.43}$$

where $\boldsymbol{f}_\kappa$ and h_κ are the force and the normal heat flux per unit area acting on the fixed boundary $\partial\mathcal{B}_\kappa$, respectively. These are the usual boundary conditions in elasticity and heat conduction.

In case the traction boundary condition is prescribed in the deformed configuration $\partial\mathcal{B}_t$, instead of in the reference configuration $\partial\mathcal{B}_\kappa$, we need to pull back the boundary force $\boldsymbol{f}(\boldsymbol{x}, t)$ to $\boldsymbol{f}_\kappa(\boldsymbol{X}, t)$. In deed, we have

$$\int_{\mathcal{S}} \boldsymbol{f}\, da = \int_{\mathcal{S}_\kappa} \boldsymbol{f}_\kappa da_\kappa, \qquad \boldsymbol{f}_\kappa = \frac{da}{da_\kappa}\, \boldsymbol{f},$$

for $\mathcal{S}_\kappa \subset \partial\mathcal{B}_\kappa$ and the corresponding $\mathcal{S} \subset \partial\mathcal{B}_t$.

In order to obtain an expression for the area ratio, we start with the relation (1.8)

$$\boldsymbol{n}\, da = J F^{-T} \boldsymbol{n}_\kappa da_\kappa,$$

from which, it follows

$$\boldsymbol{n} = \frac{F^{-T}\boldsymbol{n}_\kappa}{|F^{-T}\boldsymbol{n}_\kappa|}, \qquad \boldsymbol{n}_\kappa = \frac{F^T\boldsymbol{n}}{|F^T\boldsymbol{n}|}, \tag{2.44}$$

and with $(2.44)_2$ the above relation becomes

$$\boldsymbol{n}\, da = \frac{J}{|F^T\boldsymbol{n}|} \boldsymbol{n}\, da_\kappa.$$

Therefore, we have

$$da = \frac{J}{|F^T\boldsymbol{n}|}\, da_\kappa = \frac{\det F}{\sqrt{F^T\boldsymbol{n} \cdot F^T\boldsymbol{n}}}\, da_\kappa.$$

Similarly, with $(2.44)_1$, we have

$$da = J\, |F^{-T}\boldsymbol{n}_\kappa|\, da_\kappa = (\det F)\sqrt{F^{-T}\boldsymbol{n}_\kappa \cdot F^{-T}\boldsymbol{n}_\kappa}\; da_\kappa.$$

and hence, we obtain

$$\frac{da}{da_\kappa} = \frac{\det F}{\sqrt{\boldsymbol{n} \cdot B\boldsymbol{n}}} = (\det F)\sqrt{\boldsymbol{n}_\kappa \cdot C^{-1}\boldsymbol{n}_\kappa}, \tag{2.45}$$

where B and C are the left and the right Cauchy–Green strain tensors.

CHAPTER 3

Constitutive Relations

The balance laws introduced in Chapter 2 are the fundamental equations which are common to all material bodies. However, these laws are not sufficient to fully characterize the behavior of material bodies, because physical experiences have shown that two samples of exactly the same size and shape in general will not have the same behavior when they are subjected to exactly the same experiment (external supplies and boundary conditions).

Mathematically, the physical properties of a body can be given by a description of constitutive relations. These relations characterize the material properties of the body. It is usually assumed that such properties are intrinsic to the material body and, therefore, must be independent of external supplies as well as of different *observers* that happen to be measuring the material response simultaneously.

By an observer we mean a frame of reference for the event world and therefore is able to measure the position in the Euclidean space and time on the real line. The configuration of a body introduced so far should have been defined relative to a given frame of reference, but since only a given frame is involved, the concept of frame has been set aside until now when a change of frame of reference becomes essential in later discussions.

3.1 Frame of reference, observer

The space-time $\mathcal{W}$ is a four-dimensional space in which events occur at some places and certain instants. Let $\mathcal{W}_t$ be the space of placement of events at the instant t, then in Newtonian space-time of classical mechanics, we can write

$$\mathcal{W} = \bigcup_{t \in I\!R} \mathcal{W}_t,$$

and regard $\mathcal{W}$ as a product space $\mathcal{E} \times I\!R$, where $\mathcal{E}$ is a three dimensional Euclidean space and $I\!R$ is the space of real numbers, through a one-to one mapping, called a *frame of reference* (see [21]),

$$\phi : \mathcal{W} \to \mathcal{E} \times I\!R,$$

such that it gives rise to a one-to-one mapping for the space of placement into the Euclidean space,

$$\phi_t : \mathcal{W}_t \to \mathcal{E}. \tag{3.1}$$

The frame of reference ϕ can usually be referred to as an *observer*, since it can be depicted as taking a snapshot of events at some instant t with a camera, so that the image of ϕ_t is a picture (three-dimensional conceptually) of the placement of events, from which the distance between different simultaneous events can be measured physically, with a ruler for example. Similarly, an observer can record a sequence of snapshots at different instants with a video camera for the change of events in time.

Motion of a body

The motion of a body can be viewed as a sequence of events such that at any instant t, the placement of the body $\mathcal{B}$ in $\mathcal{W}_t$ is a one-to-one mapping

$$\chi_t : \mathcal{B} \to \mathcal{W}_t,$$

which relative to a frame of reference $\phi_t : \mathcal{W}_t \to \mathcal{E}$ can be described as

$$\chi_{\phi_t} : \mathcal{B} \to \mathcal{E}, \qquad \chi_{\phi_t} := \phi_t \circ \chi_t, \qquad \boldsymbol{x} = \chi_{\phi_t}(\boldsymbol{p}) \quad \forall\, \boldsymbol{p} \in \mathcal{B},$$

which identifies the body with a region in the Euclidean space. We call χ_{ϕ_t} a *configuration* of $\mathcal{B}$ at the instant t in the frame ϕ, and a motion of $\mathcal{B}$ is a sequence of configurations of $\mathcal{B}$ in time,

$$\chi_\phi = \{\chi_{\phi_t}, t \in I\!R \mid \chi_{\phi_t} : \mathcal{B} \to \mathcal{E}\}.$$

Reference configuration

As a primitive concept, a body $\mathcal{B}$ is considered to be a set of material points without additional mathematical structures. Although it is possible to endow the body as a manifold with a differentiable structure and topology,[1] for doing mathematics on the body, usually a *reference configuration*, say

$$\kappa_\phi : \mathcal{B} \to \mathcal{E}, \qquad \boldsymbol{X} = \kappa_\phi(\boldsymbol{p}), \qquad \mathcal{B}_{\kappa_\phi} := \kappa_\phi(\mathcal{B}),$$

is chosen, so that the motion can be treated as a function defined on the Euclidean space $\mathcal{E}$,

$$\chi_{\kappa_\phi}(\cdot, t) : \mathcal{B}_{\kappa_\phi} \subset \mathcal{E} \to \mathcal{E}, \qquad \boldsymbol{x} = \chi_{\kappa_\phi}(\boldsymbol{X}, t) \quad \forall\, \boldsymbol{X} \in \mathcal{B}_{\kappa_\phi}.$$

For examples, in previous chapters, we have already introduced

$$\dot{\boldsymbol{x}}(\boldsymbol{X}, t) = \frac{\partial \chi_{\kappa_\phi}(\boldsymbol{X}, t)}{\partial t}, \qquad \ddot{\boldsymbol{x}}(\boldsymbol{X}, t) = \frac{\partial^2 \chi_{\kappa_\phi}(\boldsymbol{X}, t)}{\partial t^2},$$

and

$$F_\kappa(\boldsymbol{X}, t) = \nabla_{\boldsymbol{X}} \chi_{\kappa_\phi}(\boldsymbol{X}, t),$$

called the velocity, the acceleration, and the deformation gradient of the motion, respectively. Note that no explicit reference to κ_ϕ is indicated in these quantities for brevity, unless similar notations reference to another frames are involved.

Remember that a configuration is a placement of a body relative to an observer, therefore, for the reference configuration κ_ϕ, there is some instant, say t_0 (not explicitly

[1] Mathematical treatment of a body as a differentiable manifold is beyond the scope of the present consideration (see [26]).

indicated for brevity), at which the reference placement of the body is made. On the other hand, the choice of a reference configuration is arbitrary, and it is not necessary that the body should actually occupy the reference place in its motion under consideration. Nevertheless, in most practical problems, t_0 is usually taken as the initial time of the motion.

3.2 Change of frame and objective tensors

Now suppose that two observers, ϕ and ϕ^*, are recording the same events with video cameras. In order to be able to compare the pictures from different viewpoints of their recordings, two observers must have a mutual agreement that the clocks of their camera must be synchronized so that simultaneous events can be recognized, and, since during the recording two people may move independently while taking pictures with different angles from their camera, there will be a relative motion and a relative orientation between them. These facts, can be expressed mathematically as follows:

We call $* := \phi^* \circ \phi^{-1}$,

$$* : \mathcal{E} \times I\!R \to \mathcal{E} \times I\!R, \qquad *(\boldsymbol{x}, t) = (\boldsymbol{x}^*, t^*),$$

a *change of frame* (observer), where $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ are the position and time of the simultaneous events observed by ϕ and ϕ^* respectively. From the observers' agreement, they must be related in the following manner,

$$\begin{aligned} \boldsymbol{x}^* &= Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}(t) \\ t^* &= t + a, \end{aligned} \tag{3.2}$$

for some $\boldsymbol{x}_0, \boldsymbol{c}(t) \in \mathcal{E}$, $Q(t) \in \mathcal{O}(V)$, $a \in I\!R$, where $\mathcal{O}(V)$ is the group of orthogonal transformations on the translation space V of the Euclidean space $\mathcal{E}$. In other words, a change of frame is an isometry of space and time as well as preserves the sense of time. Such a transformation will be called a *Euclidean transformation.*

In particular, $\phi_t^* \circ \phi_t^{-1} : \mathcal{E} \to \mathcal{E}$ is given by

$$\boldsymbol{x}^* = \phi_t^*(\phi_t^{-1}(\boldsymbol{x})) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_o) + \boldsymbol{c}(t), \tag{3.3}$$

which is a time-dependent rigid transformation consisting of an orthogonal transformation and a translation. We shall often call $Q(t)$ as the *orthogonal part* of the change of frame from ϕ to ϕ^*.

Objective tensors

The change of frame (3.2) on the Euclidean space $\mathcal{E}$ gives rise to a linear mapping on the translation space V, in the following way: Let $\boldsymbol{u}(\phi) = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in V$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{E}$ in the frame ϕ, and $\boldsymbol{u}(\phi^*) = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in V$ be the corresponding difference vector in the frame ϕ^*, then from $(3.2)_1$, it follows immediately that

$$\boldsymbol{u}(\phi^*) = Q(t)\boldsymbol{u}(\phi).$$

Any vector quantity in V, which has this transformation property, is said to be objective with respect to Euclidean transformations. The Euclidean objectivity can be generalized to any tensor spaces of V. In particular, we have the following definition.

Definition. *Let s, $\boldsymbol{u}$, and T be scalar-, vector-, (second order) tensor-valued functions respectively. If relative to a change of frame from ϕ to ϕ^* given by any Euclidean transformation,*

$$\begin{aligned}
&s(\phi^*) = s(\phi), \\
&\boldsymbol{u}(\phi^*) = Q(t)\,\boldsymbol{u}(\phi), \\
&T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^T,
\end{aligned}$$

where $Q(t)$ is the orthogonal part of the change of frame, then s, $\boldsymbol{u}$ and T are called objective *scalar, vector and tensor quantities respectively. They are also said to be frame-indifferent (in the change of frame with Euclidean transformation) or simply Euclidean objective.*

We call $f(\phi)$ the value of the function observed in the frame of reference ϕ, and for simplicity, write $f = f(\phi)$ and $f^* = f(\phi^*)$.

The definition of objective scalar is self-evident, while for objective tensors, the definition becomes obvious if one considers a simple tensor $T = \boldsymbol{u} \otimes \boldsymbol{v}$, by defining $(\boldsymbol{u} \otimes \boldsymbol{v})(\phi) = \boldsymbol{u}(\phi) \otimes \boldsymbol{v}(\phi)$ and hence

$$\begin{aligned}
(\boldsymbol{u} \otimes \boldsymbol{v})(\phi^*) &= \boldsymbol{u}(\phi^*) \otimes \boldsymbol{v}(\phi^*) \\
&= Q(t)\boldsymbol{u}(\phi) \otimes Q(t)\boldsymbol{v}(\phi) = Q(t)((\boldsymbol{u} \otimes \boldsymbol{v})(\phi))Q(t)^T,
\end{aligned}$$

as the relation between the corresponding linear transformations observed by the two observers.

One can easily deduce the transformation properties of functions defined on the position and time. Consider an objective scalar field $\psi(\boldsymbol{x}, t) = \psi^*(\boldsymbol{x}^*, t^*)$. Taking the gradient with respect to $\boldsymbol{x}$, from (3.3) we obtain

$$\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x}, t) = Q(t)^T \nabla_{\boldsymbol{x}^*}\psi^*(\boldsymbol{x}^*, t^*),$$

or

$$(\operatorname{grad}\psi)(\phi^*) = Q(t)(\operatorname{grad}\psi)(\phi), \tag{3.4}$$

which proves that $\operatorname{grad}\psi$ is an objective vector field. Similarly, we can show that if $\boldsymbol{u}$ is an objective vector field then $\operatorname{grad}\boldsymbol{u}$ is an objective tensor field. However, one can easily show that the partial derivative $\partial\psi/\partial t$ is not an objective scalar field and neither is $\partial\boldsymbol{u}/\partial t$ an objective vector field.

Proposition. *Let a surface be given by $f(\boldsymbol{x}, t) = 0$, and*

$$\boldsymbol{n}(\boldsymbol{x}, t) = \frac{\nabla_{\boldsymbol{x}} f}{|\nabla_{\boldsymbol{x}} f|}$$

be the unit normal to the surface. Then $\boldsymbol{n}$ is an objective vector field.

Proof. In the change of frame, we can write for the surface,

$$f^*(\boldsymbol{x}^*, t^*) = f^*(\boldsymbol{x}^*(\boldsymbol{x}, t), t^*(t)) = f(\boldsymbol{x}, t) = 0.$$

By taking gradient with respect to $\boldsymbol{x}$, we obtain, from (3.4) that

$$Q^T \nabla_{\boldsymbol{x}^*} f^* = \nabla_{\boldsymbol{x}} f,$$

from which, since Q is orthogonal, $|Q^T \nabla_{\boldsymbol{x}^*} f^*| = |\nabla_{\boldsymbol{x}^*} f^*|$, it follows that

$$\boldsymbol{n}^* = Q\boldsymbol{n}. \tag{3.5}$$

□

Transformation properties of motion

Let χ_ϕ be a motion of the body in the frame ϕ, and χ_{ϕ^*} be the corresponding motion in ϕ^*,

$$\boldsymbol{x} = \chi_{\phi_t}(\boldsymbol{p}) = \chi(\boldsymbol{p}, t), \qquad \boldsymbol{x}^* = \chi_{\phi_t^*}(\boldsymbol{p}) = \chi^*(\boldsymbol{p}, t^*), \qquad \boldsymbol{p} \in \mathcal{B}.$$

Then from (3.3), we have

$$\chi^*(\boldsymbol{p}, t^*) = Q(t)(\chi(\boldsymbol{p}, t) - \boldsymbol{x}_o) + \boldsymbol{c}(t), \qquad \boldsymbol{p} \in \mathcal{B}. \tag{3.6}$$

Consequently, the transformation property of a kinematic quantity, associated with the motion in $\mathcal{E}$, can be derived from (3.6). In particular, one can easily show that the velocity and the acceleration are not objective quantities. Indeed, it follows from (3.6) that

$$\begin{aligned} \dot{\boldsymbol{x}}^* &= Q\dot{\boldsymbol{x}} + \dot{Q}(\boldsymbol{x} - \boldsymbol{x}_o) + \dot{\boldsymbol{c}}, \\ \ddot{\boldsymbol{x}}^* &= Q\ddot{\boldsymbol{x}} + 2\dot{Q}\dot{\boldsymbol{x}} + \ddot{Q}(\boldsymbol{x} - \boldsymbol{x}_0) + \ddot{\boldsymbol{c}}. \end{aligned} \tag{3.7}$$

The relation $(3.7)_2$ shows that $\ddot{\boldsymbol{x}}$ is not objective but it also shows that $\ddot{\boldsymbol{x}}$ is an objective tensor with respect to transformations for which $Q(t)$ is a constant tensor. A change of frame with constant $Q(t)$ and $\boldsymbol{c}(t)$ linear in time t is called a *Galilean transformation*. Therefore, we conclude that the acceleration is a Galilean objective tensor quantity but not a Euclidean objective one. However, from $(3.7)_1$, it shows that the velocity is neither a Euclidean nor a Galilean objective vector quantity.

Transformation properties of deformation

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0, then

$$\kappa_\phi = \phi_{t_0} \circ \kappa \qquad \text{and} \qquad \kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa \tag{3.8}$$

are the two corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant (see Fig. 3.1), and

$$\boldsymbol{X} = \kappa_\phi(\boldsymbol{p}), \qquad \boldsymbol{X}^* = \kappa_{\phi^*}(\boldsymbol{p}), \qquad \boldsymbol{p} \in \mathcal{B}.$$

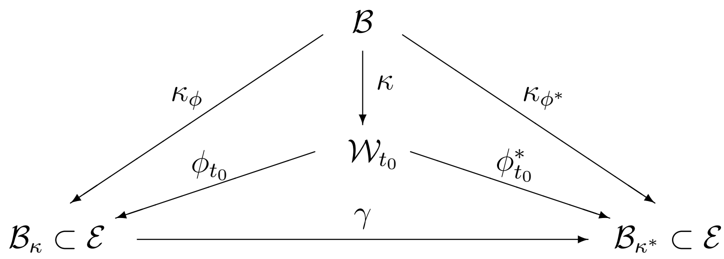

Figure 3.1: Reference configurations κ_ϕ and κ_{ϕ^*} in the change of frame from ϕ to ϕ^*

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows from (3.8) that $\gamma = \phi^*_{t_0} \circ \phi_{t_0}^{-1}$ and by (3.3), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}(t_0), \tag{3.9}$$

where $K = Q(t_0)$ is a constant orthogonal tensor.

On the other hand, the motion in referential description relative to the change of frame is given by

$$\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t), \qquad \boldsymbol{x}^* = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*),$$

and from (3.3) we have

$$\chi_{\kappa^*}(\boldsymbol{X}^*, t^*) = Q(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_o) + \boldsymbol{c}(t).$$

Therefore we obtain for the deformation gradient in the frame ϕ^*, i.e., $F^* = \nabla_{\boldsymbol{X}^*}\chi_{\kappa^*}$, by the chain rule,

$$F^*(\boldsymbol{X}^*, t^*) = Q(t)F(\boldsymbol{X}, t)\,K^T,$$

or simply,

$$F^* = QFK^T, \tag{3.10}$$

where, by (3.9), $K = Q(t_0)$, is a constant orthogonal tensor due to the change of frame for the reference configuration.[2]

The deformation gradient F is not a Euclidean objective tensor. However, the property (3.10) also shows that it is frame-indifferent with respect to Galilean transformations, since in this case, $K = Q$ is a constant orthogonal transformation.

Now we consider transformation properties of some other kinematic quantities related to the deformation gradient. With polar decompositions of F and F^*, (3.10) gives

$$R^*U^* = QRUK^T, \qquad V^*R^* = QVRK^T.$$

By the uniqueness of polar decompositions, we conclude that

$$U^* = KUK^T, \quad V^* = QVQ^T, \quad R^* = QRK^T, \tag{3.11}$$

[2]The transformation property (3.10) stands in contrast to $F^* = QF$, the well-known formula which is obtained "provided that the reference configuration is unaffected by the change of frame" (see p. 308 of [22]), so that K reduces to the identity transformation.

and also

$$C^* = KCK^T, \qquad B^* = QBQ^T. \tag{3.12}$$

Therefore, R, U, and C are not objective tensors, but the tensors V and B are objective.

If we differentiate (3.10) with respect to time, we obtain

$$\dot{F}^* = Q\dot{F}K^T + \dot{Q}FK^T.$$

With $L = \text{grad}\,\dot{\boldsymbol{x}} = \dot{F}\,F^{-1}$ by (1.24), we have

$$L^*F^* = QLFK^T + \dot{Q}FK^T = QLQ^TF^* + \dot{Q}Q^TF^*,$$

and since F^* is non-singular, it gives

$$L^* = QLQ^T + \dot{Q}Q^T. \tag{3.13}$$

Moreover, with $L = D + W$, it becomes

$$D^* + W^* = Q(D + W)Q^T + \dot{Q}Q^T.$$

By separating symmetric and skew-symmetric parts, we obtain

$$D^* = QDQ^T, \qquad W^* = QWQ^T + \dot{Q}Q^T, \tag{3.14}$$

since $\dot{Q}Q^T$ is skew-symmetric. Therefore, while the velocity gradient $\text{grad}\,\dot{\boldsymbol{x}}$ and the spin tensor W are not objective, the rate of strain tensor D is an objective tensor.

Objective time derivatives

Let ψ and $\boldsymbol{u}$ be objective scalar and vector fields respectively,

$$\psi^*(\boldsymbol{X}^*, t^*) = \psi(\boldsymbol{X}, t), \qquad \boldsymbol{u}^*(\boldsymbol{X}^*, t^*) = Q(t)\boldsymbol{u}(\boldsymbol{X}, t).$$

It follows that

$$\dot{\psi}^* = \dot{\psi}, \qquad \dot{\boldsymbol{u}}^* = Q\dot{\boldsymbol{u}} + \dot{Q}\boldsymbol{u}.$$

Therefore, the material time derivative of an objective scalar field is objective, while the material time derivative of an objective vector field is not longer objective.

Nevertheless, we can define an objective time derivatives of vector field in the following way: Let $P_t(\tau) : V \to V$ be a transformation which takes a vector at time t to a vector at time τ and define the time derivative as

$$D_{P_t}\boldsymbol{u}(t) = \lim_{h\to 0} \frac{1}{h}(\boldsymbol{u}(t+h) - P_t(t+h)\boldsymbol{u}(t)), \tag{3.15}$$

so that the vector $\boldsymbol{u}(t)$ is transformed to a vector at $(t+h)$ through $P_t(t+h)$ in order to be compared with the vector $\boldsymbol{u}(t+h)$ at the same instant of time. This is usually called a Lie derivative.

Now, we shall take $P_t(\tau)$ to be the relative rotation tensor of the motion $R_t(\tau)$ (see (1.30)) and denote

$$\overset{\circ}{\boldsymbol{u}} = D_{R_t}\boldsymbol{u}.$$

Since

$$R_t(t+h) = R_t(t) + \dot{R}_t(t)h + o(2)$$

and from (1.32)

$$R_t(t) = 1, \qquad \dot{R}_t(t) = W(t),$$

we have

$$\overset{\circ}{\boldsymbol{u}}(t) = \lim_{h\to 0} \frac{1}{h}(\boldsymbol{u}(t+h) - \boldsymbol{u}(t) - h\,W(t)\boldsymbol{u}(t)),$$

or

$$\overset{\circ}{\boldsymbol{u}} = \dot{\boldsymbol{u}} - W\boldsymbol{u}. \tag{3.16}$$

Note that by $(3.14)_2$,

$$\begin{aligned}(\overset{\circ}{\boldsymbol{u}})^* &= \dot{\boldsymbol{u}}^* - W^*\boldsymbol{u}^* \\ &= Q\dot{\boldsymbol{u}} + \dot{Q}\boldsymbol{u} - (QWQ^T + \dot{Q}Q^T)Q\boldsymbol{u} \\ &= Q(\dot{\boldsymbol{u}} - W\boldsymbol{u}) = Q\,\overset{\circ}{\boldsymbol{u}}\,.\end{aligned}$$

Therefore, $\overset{\circ}{\boldsymbol{u}}$ is an objective time derivative of $\boldsymbol{u}$. This derivative is called the *corotational time derivative*, which measures the time rate of change experienced by material particles rotating along with the motion.

Similarly, if we take $P_t(\tau)$ to be the relative deformation gradient of the motion $F_t(\tau)$ (see (1.30)) and denote

$$\overset{\triangle}{\boldsymbol{u}} = D_{F_t}\boldsymbol{u}.$$

It is an called the *convected time derivatives*, which measures the time rate of change experienced by material particles carried along with the motion. It follows that

$$\overset{\triangle}{\boldsymbol{u}} = \dot{\boldsymbol{u}} - L\boldsymbol{u},$$

which is also an objective time derivative.

For an objective tensor field S, we can define

$$D_{P_t}S(t) = \lim_{h\to 0}\frac{1}{h}(S(t+h) - P_t(t+h)S(t)P_t(t+h)^T), \tag{3.17}$$

and let

$$\overset{\circ}{S} = D_{R_t}S, \qquad \overset{\triangle}{S} = D_{F_t}S,$$

which define the objective corotational and convective time derivative of an objective tensor field S as

$$\begin{aligned}\overset{\circ}{S} &= \dot{S} - WS + SW, \\ \overset{\triangle}{S} &= \dot{S} - LS - SL^T,\end{aligned}$$

respectively.

3.3 Constitutive relations

Physically the thermomechanical behavior of a body is characterized by a description of the fields of density $\rho(\boldsymbol{p}, t)$, motion $\chi(\boldsymbol{p}, t)$ and temperature $\theta(\boldsymbol{p}, t)$, called the basic thermodynamic fields. The material response of a body generally depends on the past history of its thermomechanical behavior.

Let us introduce the notion of the past history of a function. Let $h(\cdot)$ be a function of time. The *history* of h up to time t is defined by

$$h^t(s) = h(t - s), \tag{3.18}$$

where $s \in [0, \infty)$ denotes the time-coordinate pointed into the past from the present time t. Clearly $s = 0$ corresponds to the present time, therefore

$$h^t(0) = h(t).$$

Mathematical descriptions of material response are called *constitutive relations* or constitutive equations. We postulate that the history of the behavior up to the present time determines the response of the body.

Principle of determinism (in material description)**.** *Let ϕ be a frame of reference, and $\mathcal{C}$ be a constitutive quantity, then the constitutive relation for $\mathcal{C}$ is given by a functional of the form,*[3]

$$\mathcal{C}(\boldsymbol{p}, t) = \underset{\substack{\boldsymbol{p}' \in \mathcal{B} \\ 0 \le s < \infty}}{\mathcal{F}_\phi} \ (\rho^t(\boldsymbol{p}', s), \chi^t(\boldsymbol{p}', s), \theta^t(\boldsymbol{p}', s), t\,; \boldsymbol{p}), \quad \boldsymbol{p} \in \mathcal{B},\ t \in I\!R. \tag{3.19}$$

We call $\mathcal{F}_\phi$ the *constitutive function* or *response function* of $\mathcal{C}$ in the frame ϕ. Note that we have indicated the domains of the argument functions as underscripts in the notation of the functional $\mathcal{F}_\phi$. Such a functional allows the description of arbitrary non-local effect of any inhomogeneous body with a perfect memory of the past thermomechanical history. With the notation $\mathcal{F}_\phi$, we emphasize that the value of a constitutive function *a priori* may depend on the observer (frame of reference ϕ) who is measuring the response of the material body and the material point $\boldsymbol{p}$.

Constitutive relations can be regarded as mathematical models of material bodies. The validity of a model can be verified by experiments. However, experiments alone are rarely sufficient to determine constitutive functions of a material body. Nevertheless, there are some universal requirements that a model should obey lest its consequences be contradictory to some well-known physical experiences. Therefore, in search of a correct

[3] The functional $\mathcal{F}_\phi$ can be expressed more directly as

$$\mathcal{C}(\boldsymbol{p}, t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t\,; \boldsymbol{p}),$$

where the first three arguments are functions:

$$\rho^t : \mathcal{B} \times [0, \infty) \to I\!R, \qquad \chi^t : \mathcal{B} \times [0, \infty) \to \mathcal{E}, \qquad \theta^t : \mathcal{B} \times [0, \infty) \to I\!R.$$

formulation of a mathematical model in general, we shall first impose these requirements on the proposed model. The most important universal requirements of this kind are

– *Euclidean objectivity,*
– *principle of material frame-indifference,*
– *material symmetry,*
– *thermodynamic considerations.*

Euclidean objectivity, principle of material frame-indifference and material symmetry will be discussed in this chapter and a brief account of thermodynamic considerations will be given later. These requirements impose severe restrictions on the model and hence lead to a great simplification for general constitutive relations. The reduction of constitutive relations from very general to more specific and mathematically simpler ones for a given class of materials is the main objective of constitutive theories in continuum mechanics.

3.4 Euclidean objectivity

In addition to the conservation laws in an inertial frame stated in the previous chapter, we *should* have also postulated that the mass density, the internal energy (but not the total energy) and the temperature are objective scalar quantities, while the heat flux and the surface traction are objective vector quantities, so that the conservation laws relative to an arbitrary frame of reference can be deduced (see [14]).

In particular, under a change of frame, we have $\boldsymbol{t}^* = Q\boldsymbol{t}$ for the surface traction and since $\boldsymbol{t} = T\boldsymbol{n}$, and $\boldsymbol{n}^* = Q\boldsymbol{n}$ from (3.5), it follows that the stress T is an objective tensor quantity. To emphasize the objectivity postulate in the conservation laws, it is referred to as *Euclidean objectivity* of constitutive quantities:

Euclidean objectivity. *The constitutive quantities: the stress T, the heat flux $\boldsymbol{q}$ and the internal energy density ε, are objective with respect to any Euclidean change of frame,*

$$\begin{aligned} &T^*(\boldsymbol{p},t^*) = Q(t)\,T(\boldsymbol{p},t)\,Q(t)^T,\\ &\boldsymbol{q}^*(\boldsymbol{p},t^*) = Q(t)\,\boldsymbol{q}(\boldsymbol{p},t),\\ &\varepsilon^*(\boldsymbol{p},t^*) = \varepsilon(\boldsymbol{p},t), \end{aligned} \tag{3.20}$$

where $Q(t)$ is the orthogonal part of the change of frame.

Note that this postulate is a universal requirement in order to formulate the conservation laws of material bodies in an arbitrary frame of reference, and no constitutive assumptions are involved in the postulate of Euclidean objectivity.[4]

Let $\mathcal{C} = \{T, \boldsymbol{q}, \varepsilon\}$ be constitutive quantities and ϕ be a frame of reference. Then from (3.19), the constitutive relations can be written as

$$\mathcal{C}(\boldsymbol{p},t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t\,;\boldsymbol{p}), \qquad \boldsymbol{p}\in\mathcal{B},\ t\in I\!R, \tag{3.21}$$

[4]In early writings of Truesdell and Noll (see [22]), this postulate was stated as part of the principle of material frame-indifference. It would be much clear conceptually, if it were stated independently, because constitutive equations are irrelevant in this assumption.

where $\mathcal{F}_\phi$ is a functional with argument functions ρ^t, χ^t, θ^t, defined on $\mathcal{B}\times[0,\infty)$, stand for the past histories up to the instant t. Similarly, relative to the frame ϕ^*, the corresponding constitutive relation can be written as

$$\mathcal{C}^*(\boldsymbol{p}, t^*) = \mathcal{F}_{\phi^*}\Big((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*; \boldsymbol{p}\Big), \qquad \boldsymbol{p} \in \mathcal{B},\ t^* \in I\!R. \tag{3.22}$$

The two constitutive functions $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi^*}$ are not independent, they must satisfy the Euclidean objectivity relation (3.20). In particular, for the stress, the condition (3.20) implies that

$$\mathcal{F}_{\phi^*}((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*; \boldsymbol{p}) = Q(t)\, \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t; \boldsymbol{p})\, Q(t)^T, \tag{3.23}$$

for any histories

$$\begin{aligned} &(\rho^t)^*(\boldsymbol{p}', s) = \rho^*(\boldsymbol{p}', t^* - s) = \rho(\boldsymbol{p}', t - s), \\ &(\theta^t)^*(\boldsymbol{p}', s) = \theta^*(\boldsymbol{p}', t^* - s) = \theta(\boldsymbol{p}', t - s), \qquad\qquad \boldsymbol{p}' \in \mathcal{B}, \\ &(\chi^t)^*(\boldsymbol{p}', s) = Q(t-s)(\chi^t(\boldsymbol{p}', s) - \boldsymbol{x}_o) + \boldsymbol{c}(t - s), \end{aligned} \tag{3.24}$$

where $Q(t) \in \mathcal{O}(V)$, $\boldsymbol{x}_o, \boldsymbol{c}(t) \in \mathcal{E}$ are associated with the change of frame from ϕ to ϕ^*.

The first two relations of (3.24) state that the density and the temperature are objective scalar field as assumed and the last relation follows from (3.3).

The relation (3.23) will be referred to as the “condition of Euclidean objectivity” of constitutive functions, and can be regarded as the *definition* of the $\mathcal{F}_{\phi^*}$ once the constitutive function $\mathcal{F}_\phi$ is given[5].

3.5 Principle of material frame indifference

The essential idea of the principle of material frame-indifference is that material properties are frame-indifferent, i.e., material properties must be independent of observers. This statement only makes sense when applies to constitutive quantities that are themselves frame-indifferent, i.e., they must be objective quantities.

Since any intrinsic property of a material should be independent of observers, we postulate that for any objective constitutive quantity, its constitutive function must be the same in any frame.

Principle of material frame-indifference. *The response function of an objective (frame-indifferent with respect to the Euclidean transformations) constitutive quantity $\mathcal{C}$, in material description defined by (3.21) and (3.22), must be independent of the frame, i.e., for any frames of reference ϕ and ϕ^**

$$\mathcal{F}_\phi(\,\bullet\,; \boldsymbol{p}) = \mathcal{F}_{\phi^*}(\,\bullet\,; \boldsymbol{p}), \qquad \boldsymbol{p} \in \mathcal{B}. \tag{3.25}$$

[5] For example, consider $T = \mathcal{F}_\phi(\rho, L)$. Since by Euclidean objectivity, $\rho^* = \rho$, $T^* = QTQ^T$, and from (3.13), $L^* = QLQ^T + \dot{Q}Q^T$, it follows that

$$\begin{aligned} T^* &= Q\,\mathcal{F}_\phi(\rho, L)\,Q^T \\ &= Q\,\mathcal{F}_\phi(\rho^*, Q^T L^* Q - Q^T\dot{Q})\,Q^T := \widehat{\mathcal{F}}_\phi(\rho^*, L^*; Q, \dot{Q}) := \mathcal{F}_{\phi^*}(\rho^*, L^*), \end{aligned}$$

where $\mathcal{F}_{\phi^*}$, defined in the above relation, depends explicitly on $Q(t)$ of the change of frame $\phi \to \phi^*$.

We emphasize that the "form-invariance" (3.25) of constitutive function stated above is valid only when the *material description* is used so that the constitutive functions characterize the material properties independent of any reference configuration. The implication in referential descriptions will be considered later. Moreover, for a non-objective constitutive quantity, such as the total energy, since the quantity itself is frame-dependent, it is obvious that its response function can never be independent of the frame.

In our discussions, we shall often consider only the constitutive function of stress, an objective tensor quantity, for simplicity. Similar results can easily be obtained for any other vector or scalar objective constitutive quantities.

Thus, from the condition of Euclidean objectivity (3.23) and the principle of material frame-indifference (3.25), we obtain the following condition:

Condition of material objectivity. *The response function of an objective tensor constitutive quantity, in material description, satisfies the condition,*

$$\mathcal{F}_\phi((\rho^t)^*, (\chi^t)^*, (\theta^t)^*, t^*, \boldsymbol{p}) = Q(t)\, \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t\,; \boldsymbol{p})\, Q(t)^T, \tag{3.26}$$

for any histories related by (3.24) where the change of frame from ϕ is arbitrary.

Since the condition (3.26) involves only the constitutive function in the frame ϕ, it becomes a restriction imposed on the constitutive function $\mathcal{F}_\phi$.

We emphasize that in the condition of Euclidean objectivity (3.23), $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*. However, the condition of material objectivity is valid for an arbitrary change of frame from ϕ. Therefore, the condition (3.26) is valid for any $Q(t) \in \mathcal{O}(V)$.

Sometimes, the *condition* of material objectivity is referred to as the "*principle* of material objectivity", to impart its relevance in characterizing material property and Euclidean objectivity, as a more explicit form of the principle of material-frame indifference.

An immediate restriction imposed by the condition of material objectivity can be obtained by considering a change of frame given by ($Q(t) = 1$, $\boldsymbol{c}(t) = \boldsymbol{x}_0$)

$$\boldsymbol{x}^* = \boldsymbol{x}, \qquad t^* = t + a,$$

for arbitrary constant $a \in I\!R$. By (3.24), the condition (3.26) implies that

$$\mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t + a\,; \boldsymbol{p}) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t\,; \boldsymbol{p}).$$

Since this is true for any value of $a \in I\!R$, we conclude that $\mathcal{F}_\phi$ can not depend on the argument t and the constitutive equations in general can be expressed as

$$\mathcal{C}(\boldsymbol{p}, t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t; \boldsymbol{p}), \qquad \boldsymbol{p} \in \mathcal{B}, \quad t \in I\!R.$$

Conditions in referential description

Now let us consider the implication of form-invariance and the condition of material objectivity in referential description. Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0, then $\kappa_\phi = \phi_{t_0} \circ \kappa : \mathcal{B} \to \mathcal{E}$ and $\kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa : \mathcal{B} \to \mathcal{E}$ are the two corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant (see Fig. 3.1), and

$$\boldsymbol{X} = \kappa_\phi(\boldsymbol{p}) \in \mathcal{E}, \qquad \boldsymbol{X}^* = \kappa_{\phi^*}(\boldsymbol{p}) \in \mathcal{E}, \qquad \boldsymbol{p} \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows that $\gamma = \phi^*_{t_0} \circ \phi^{-1}_{t_0}$ and by (3.3), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}(t_0), \tag{3.27}$$

where $K = \nabla_{\boldsymbol{X}}\gamma = Q(t_0)$ is a constant orthogonal tensor.

The motion in referential description relative to the change of frame is given by

$$\begin{aligned} \boldsymbol{x} &= \chi(\boldsymbol{p}, t) &&= \chi(\kappa_\phi^{-1}(\boldsymbol{X}), t) &&= \chi_\kappa(\boldsymbol{X}, t), && \chi = \chi_\kappa \circ \kappa_\phi, \\ \boldsymbol{x}^* &= \chi^*(\boldsymbol{p}, t^*) &&= \chi^*({\kappa_{\phi^*}}^{-1}(\boldsymbol{X}^*), t^*) &&= \chi_{\kappa^*}(\boldsymbol{X}^*, t^*), && \chi^* = \chi_{\kappa^*} \circ \kappa_{\phi^*}. \end{aligned}$$

From (3.21) and (3.22), we can define the corresponding constitutive functions with respect to the reference configuration,

$$\begin{aligned} \mathcal{F}_\phi(\chi^t; \boldsymbol{p}) &= \mathcal{F}_\phi(\chi^t_\kappa \circ \kappa_\phi; \boldsymbol{p}) &&:= \mathcal{H}_\kappa(\chi^t_\kappa; \boldsymbol{X}), \\ \mathcal{F}_{\phi^*}((\chi^t)^*; \boldsymbol{p}) &= \mathcal{F}_{\phi^*}((\chi^t_\kappa)^* \circ \kappa_{\phi^*}; \boldsymbol{p}) &&:= \mathcal{H}_{\kappa^*}((\chi^t_\kappa)^*; \boldsymbol{X}^*), \end{aligned}$$

where for simplicity, only one argument function χ^t is written out symbolically.

Since the form-invariance (3.25) requires $\mathcal{F}_\phi = \mathcal{F}_{\phi^*}$, from the definition, it implies that the constitutive functions $\mathcal{H}_\kappa$ and $\mathcal{H}_{\kappa^*}$ in the referential description are related by

$$\begin{aligned} \mathcal{H}_{\kappa^*}((\chi^t)^*; \boldsymbol{X}^*) &= \mathcal{F}_{\phi^*}((\chi^t_\kappa)^* \circ \kappa_{\phi^*}; \boldsymbol{p}) = \mathcal{F}_\phi((\chi^t_\kappa)^* \circ \kappa_{\phi^*}; \boldsymbol{p}) \\ &= \mathcal{F}_\phi((\chi^t_\kappa)^* \circ \gamma \circ \kappa_\phi; \boldsymbol{p}) = \mathcal{H}_\kappa((\chi^t)^* \circ \gamma; \boldsymbol{X}), \end{aligned}$$

where $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1} = \phi^*_{t_0} \circ \phi^{-1}_{t_0}$. Therefore, they are not form-invariant in general, i.e., $\mathcal{H}_{\kappa^*} \neq \mathcal{H}_\kappa$, but rather they are related by[6]

$$\mathcal{H}_{\kappa^*}(\,\bullet\,; \boldsymbol{X}^*) = \mathcal{H}_\kappa(\,\bullet \circ \gamma; \boldsymbol{X}), \tag{3.28}$$

The Euclidean objectivity relation (3.20) can now be written in the form,

$$\mathcal{H}_{\kappa^*}((\chi^t_\kappa)^*; \boldsymbol{X}^*) = Q(t)\, \mathcal{H}_\kappa(\chi^t_\kappa; \boldsymbol{X})\, Q(t)^T. \tag{3.29}$$

Moreover, by combining (3.28) and (3.29), we obtain the condition of material objectivity in referential description,

$$\mathcal{H}_\kappa((\chi^t_\kappa)^* \circ \gamma; \boldsymbol{X}) = Q(t)\, \mathcal{H}_\kappa(\chi^t_\kappa; \boldsymbol{X})\, Q(t)^T, \qquad \forall\, Q(t) \in \mathcal{O}(V). \tag{3.30}$$

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[6]If the reference configuration is assumed to be unaffected by the change of frame, then γ is an identity map on $\mathcal{E}$, and $\mathcal{H}$ is form-invariant, $\mathcal{H}_{\kappa^*} = \mathcal{H}_\kappa$. This was implicitly assumed in the original statement of the principle of material frame-indifference (see [22]).

3.6 Simple material bodies

According to (3.19), thermomechanical histories of any part of the body can affect the response at any point of the body. In most applications, such a non-local property is irrelevant. Therefore it is usually assumed that only thermomechanical histories in an arbitrary small neighborhood of $\boldsymbol{X} \in \mathcal{B}_\kappa$ affects the material response at the point $\boldsymbol{X}$, and hence the histories can be approximated at $\boldsymbol{X}$ by Taylor series up to certain order. In particular, when only linear approximation is concerned, in referential description the constitutive relation (3.19) can be written as

$$\mathcal{C}(\boldsymbol{X},t) = \underset{0\leq s<\infty}{\mathcal{H}_\kappa}(\chi^t_\kappa(\boldsymbol{X},s), F^t(\boldsymbol{X},s), \theta^t(\boldsymbol{X},s), \boldsymbol{g}^t(\boldsymbol{X},s); \boldsymbol{X}), \tag{3.31}$$

where $\boldsymbol{g} = \operatorname{grad}\theta$ is the spatial gradient of temperature. Note that since

$$\rho(\boldsymbol{X},t) = \frac{\rho_\kappa(\boldsymbol{X})}{|\det F(\boldsymbol{X},t)|},$$

the functional dependence of (3.19) on ρ^t is absorbed into the dependence of (3.31) on $\boldsymbol{X}$ and $F^t(\boldsymbol{X},s)$.

A material body with constitutive relation defined by (3.31) is called a *simple material body*. The class of simple materials, introduced by Noll [18], is general enough to include most of the materials of practical interests, such as: the elastic solids, thermoelastic solids, viscoelastic solids as well as elastic fluids, Navier-Stokes fluids and viscous heat-conducting fluids.

An immediate consequence of the condition of material objectivity (3.30) can be obtained by the following choices of change of frame. Consider a change of frame given by $(Q(t) = 1,\ a = 0)$

$$\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{c}(t) - \boldsymbol{x}_0, \qquad t^* = t.$$

Clearly, we have

$$\chi_{\kappa^*}(\gamma(\boldsymbol{X}),t) = \chi_\kappa(\boldsymbol{X},t) + (\boldsymbol{c}(t) - \boldsymbol{x}_0),$$

and the condition (3.30) implies that

$$\mathcal{H}_\kappa(\chi^t_\kappa + \boldsymbol{c}^t - \boldsymbol{x}_0, F^t, \boldsymbol{X}) = \mathcal{H}_\kappa(\chi^t_\kappa, F^t, \boldsymbol{X}).$$

Since $(\boldsymbol{c}^t(s) - \boldsymbol{x}_0) \in V$ is arbitrary, we conclude that $\mathcal{H}_\kappa$ can not depend on the history of position $\chi^t_\kappa(\boldsymbol{X},s)$.

Therefore, the constitutive relation of a simple material body in general is given by

$$\mathcal{C}(\boldsymbol{X},t) = \underset{0\leq s<\infty}{\mathcal{H}_\kappa}(F^t(\boldsymbol{X},s), \theta^t(\boldsymbol{X},s), \boldsymbol{g}^t(\boldsymbol{X},s); \boldsymbol{X}). \tag{3.32}$$

It is important to point out that constitutive functions can not depend on the position $\boldsymbol{x}$ nor on the velocity $\dot{\boldsymbol{x}}$, and indeed this can be proved for any materials in general as a consequence of the principle of material objectivity (3.26).

For simple materials, the consequence of form-invariance (3.28) by the use of (3.9) takes the form

$$\mathcal{H}_{\kappa^*}((F^t)^*;\boldsymbol{X}^*) = \mathcal{H}_{\kappa}((F^t)^* K;\boldsymbol{X}), \tag{3.33}$$

and the Euclidean objectivity condition (3.29) becomes

$$\mathcal{H}_{\kappa^*}((F^t)^*;\boldsymbol{X}^*) = Q(t)\,\mathcal{H}_{\kappa}(F^t;\boldsymbol{X})\,Q(t)^T.$$

Combining the above two conditions and knowing the relation $F^*K = (QFK^T)K = QF$, we obtain the following condition,[7]

$$\mathcal{H}_{\kappa}(Q^t F^t;\boldsymbol{X}) = Q(t)\,\mathcal{H}_{\kappa}(F^t;\boldsymbol{X})\,Q(t)^T. \tag{3.34}$$

More generally, we have the following conditions for objective constitutive functions of simple materials:

Condition of material objectivity. *Constitutive functions for simple materials must satisfy the following conditions, for objective tensor, vector and scalar constitutive quantities respectively,*

$$\begin{aligned} \mathcal{T}(Q^tF^t,\theta^t,Q^t\boldsymbol{g}^t;\boldsymbol{X}) &= Q\,\mathcal{T}(F^t,\theta^t,\boldsymbol{g}^t;\boldsymbol{X})\,Q^T, \\ \mathcal{Q}(Q^tF^t,\theta^t,Q^t\boldsymbol{g}^t;\boldsymbol{X}) &= Q\,\mathcal{Q}(F^t,\theta^t,\boldsymbol{g}^t;\boldsymbol{X}), \qquad \forall\, Q \in \mathcal{O}(V), \\ \mathcal{E}\,(Q^tF^t,\theta^t,Q^t\boldsymbol{g}^t;\boldsymbol{X}) &= \mathcal{E}\,(F^t,\theta^t,\boldsymbol{g}^t;\boldsymbol{X}), \end{aligned} \tag{3.35}$$

and for any thermomechanical histories $(F^t,\theta^t,\boldsymbol{g}^t)$.

These conditions are the restriction imposed on the constitutive functions by the requirement of material objectivity.

Note that in condition (3.35), no mention of change of frame is involved, and $Q(t)$ can be interpreted as a superimposed orthogonal transformation on the deformation. This interpretation is sometimes viewed as an alternative version of the principle of material objectivity, and is called the "principle of invariance under *superimposed rigid body motions*".

3.7 Material symmetry

We shall consider homogeneous simple material bodies in this section. A body is called *homogeneous* if there is a reference configuration κ such that the constitutive function (3.32) does not depend on the argument $\boldsymbol{X}$ explicitly and we can write

$$\mathcal{C}(\boldsymbol{x},t) = \mathop{\mathcal{H}_{\kappa}}_{0\le s<\infty}(F^t(\boldsymbol{X},s),\theta^t(\boldsymbol{X},s),\boldsymbol{g}^t(\boldsymbol{X},s)), \qquad \boldsymbol{x} = \chi_{\kappa}(\boldsymbol{X},t). \tag{3.36}$$

Suppose that $\hat\kappa$ is another reference configuration and the constitutive relations relative to $\hat\kappa$ can be written as

$$\mathcal{C}(\boldsymbol{x},t) = \mathop{\mathcal{H}_{\hat\kappa}}_{0\le s<\infty}(\widehat{F}^t(\hat{\boldsymbol{X}},s),\theta^t(\hat{\boldsymbol{X}},s),\boldsymbol{g}^t(\hat{\boldsymbol{X}},s)), \qquad \boldsymbol{x} = \hat\chi(\hat{\boldsymbol{X}},t). \tag{3.37}$$

[7]Note that this is the same as the well-known condition of material objectivity, obtained with the assumption that reference configuration be unaffected by the change of frame in [22] and other literatures in continuum mechanics.

Let $G = \nabla_{\boldsymbol{X}}(\hat{\kappa} \circ \kappa^{-1})$, then

$$F^t = \widehat{F}^t G.$$

By comparing (3.36) and (3.37), we have the following relation between the response functions $\mathcal{H}_\kappa$ and $\mathcal{H}_{\hat{\kappa}}$,

$$\mathcal{H}_{\hat{\kappa}}(\widehat{F}^t, \theta^t, \boldsymbol{g}^t) = \mathcal{H}_\kappa(\widehat{F}^t G, \theta^t, \boldsymbol{g}^t). \tag{3.38}$$

A material body subjected to the same thermomechanical history at two different configurations κ and $\hat{\kappa}$ may have different results. However, it may happen that the results are the same if the material possesses a certain symmetry. For example, one can not distinguish the response of a material body with a cubic crystal structure before and after a rotation of $90°$ about one of its crystal axes.

Definition. *Two reference configurations κ and $\hat{\kappa}$ are said to be materially indistinguishable if their corresponding constitutive functions are the same,*

$$\mathcal{H}_\kappa(\bullet) = \mathcal{H}_{\hat{\kappa}}(\bullet).$$

By (3.38), the above condition is equivalent to

$$\mathcal{H}_\kappa(F^t, \theta^t, \boldsymbol{g}^t) = \mathcal{H}_\kappa(F^t G, \theta^t, \boldsymbol{g}^t), \qquad \forall\, (F^t, \theta^t, \boldsymbol{g}^t). \tag{3.39}$$

We call a transformation $G \in \mathcal{L}(V)$ which satisfies (3.39) a *material symmetry transformation* with respect to κ.

We assume that a material symmetry transformation is volume-preserving. Since, otherwise, if G is a material symmetry transformation, so is G^n for any $n = 1, 2, \cdots$, and therefore, the material could suffer arbitrarily large dilatation or arbitrary contraction with no change in material response — a conclusion that seems physically unacceptable (for another justifications see [8, 11]). Therefore, we must require that $G \in \mathcal{U}(V)$, where $\mathcal{U}(V) = \{G \in \mathcal{L}(V) : |\det G| = 1\}$ is called the *unimodular group*. It is easy to verify that

Proposition. *The set of all material symmetry transformations*

$$\mathcal{G}_\kappa = \{G \in \mathcal{U}(V) \,|\, \mathcal{H}_\kappa(F^t, \theta^t, \boldsymbol{g}^t) = \mathcal{H}_\kappa(F^t G, \theta^t, \boldsymbol{g}^t),\ \forall\, F^t \in \mathcal{L}(V)\},$$

is a subgroup of the unimodular group.

We call $\mathcal{G}_\kappa$ the *material symmetry group* of the material body in the reference configuration κ. We have

Condition of material symmetry. *Constitutive functions for simple materials must satisfy the following conditions, for objective tensor, vector and scalar constitutive quantities respectively,*

$$\begin{aligned}
\mathcal{T}(F^t G, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}) &= \mathcal{T}(F^t, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}),\\
\mathcal{Q}(F^t G, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}) &= \mathcal{Q}(F^t, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}), \qquad \forall\, G \in \mathcal{G}_\kappa,\\
\mathcal{E}\,(F^t G, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}) &= \mathcal{E}\,(F^t, \theta^t, \boldsymbol{g}^t; \boldsymbol{X}),
\end{aligned} \tag{3.40}$$

and for any thermomechanical histories $(F^t, \theta^t, \boldsymbol{g}^t)$.

It is clear that the symmetry group depends on the reference configuration. Suppose that $\hat{\kappa}$ is another reference configuration such that $P = \nabla_{\boldsymbol{X}}(\hat{\kappa} \circ \kappa^{-1})$. Then for any $G \in \mathcal{G}_\kappa$, from (3.38) we have

$$\begin{aligned} \mathcal{H}_{\hat{\kappa}}(F^t) &= \mathcal{H}_\kappa(F^t P) = \mathcal{H}_\kappa(F^t P G) \\ &= \mathcal{H}_\kappa(F^t (PGP^{-1}) P) = \mathcal{H}_{\hat{\kappa}}(F^t (PGP^{-1})), \end{aligned}$$

which implies that $PGP^{-1} \in \mathcal{G}_{\hat{\kappa}}$. Therefore we have proved the following proposition.

Proposition (Noll's rule)**.** *For any κ and $\hat{\kappa}$, such that $P = \nabla_{\boldsymbol{X}}(\hat{\kappa} \circ \kappa^{-1})$, the following relation hold,*

$$\mathcal{G}_{\hat{\kappa}} = P \mathcal{G}_\kappa P^{-1}. \tag{3.41}$$

Physical concepts of real materials such as *solids* and *fluids*, are usually characterized by their symmetry properties. One of such concepts can be interpreted as saying that a solid has a preferred configuration such that any non-rigid deformation from it alters its material response, while for a fluid any deformation that preserves the density should not affect the material response. Based on this concept, we shall give the following definitions of solids and fluids due to Noll.

Definition. *A material body is called a* simple solid *body if there exists a reference configuration κ, such that $\mathcal{G}_\kappa$ is a subgroup of the orthogonal group, i.e., $\mathcal{G}_\kappa \subseteq \mathcal{O}(V)$.*

Definition. *A material is called a* simple fluid *if the symmetry group is the full unimodular group, i.e., $\mathcal{G}_\kappa = \mathcal{U}(V)$.*

For a simple fluid, the Noll's rule (3.41) implies that the symmetry group is the unimodular group relative to any configuration. In other words, a fluid does not have a preferred configuration.

A material which is neither a fluid nor a solid will be called a *fluid crystal*. In other words, for a fluid crystal, there does not exist a reference configuration κ for which either $\mathcal{G}_\kappa \subseteq \mathcal{O}(V)$ or $\mathcal{G}_\kappa = \mathcal{U}(V)$. We shall not consider fluid crystals in this note.

Definition. *A material body is called* isotropic if there exists a configuration κ, such that the symmetry group contains the orthogonal group, $\mathcal{G}_\kappa \supseteq \mathcal{O}(V)$.

Physically, we can interpret the above definition as saying that any rotation from an isotropic configuration does not alter the material response. The following theorem characterizes isotropic materials. For the proof of the theorem see [17].

Theorem. *The orthogonal group is maximal in the unimodular group, i.e., if g is a group such that*

$$\mathcal{O}(V) \subset g \subset \mathcal{U}(V),$$

then either $g = \mathcal{O}(V)$ or $g = \mathcal{U}(V)$.

Therefore, an isotropic material is either a fluid, $\mathcal{G}_\kappa = \mathcal{U}(V)$ for any κ, or an isotropic solid, $\mathcal{G}_\kappa = \mathcal{O}(V)$ for some configuration κ. Any other materials must be *anisotropic*.

Transversely isotropic solids, crystalline solids and fluid crystals are all anisotropic materials.

For an isotropic material, since $\mathcal{O}(V) \subseteq \mathcal{G}_\kappa$ with respect to an isotropic configuration, from the condition of material objectivity (3.35) and the material symmetry condition (3.40), we have

Proposition. *For an isotropic material, the constitutive functions are isotropic functions, namely, the following conditions,*

$$\begin{aligned} \mathcal{T}(QF^tQ^T, \theta^t, Q\boldsymbol{g}^t) &= Q\,\mathcal{T}(F^t, \theta^t, \boldsymbol{g}^t)\,Q^T, \\ \mathcal{Q}(QF^tQ^T, \theta^t, Q\boldsymbol{g}^t) &= Q\,\mathcal{Q}(F^t, \theta^t, \boldsymbol{g}^t), \qquad \forall\, Q \in \mathcal{O}(V), \\ \mathcal{E}\,(QF^tQ^T, \theta^t, Q\boldsymbol{g}^t) &= \mathcal{E}\,(F^t, \theta^t, \boldsymbol{g}^t), \end{aligned} \tag{3.42}$$

hold for tensor-, vector- and scalar-valued constitutive quantities respectively.

Since constitutive functions are, in general, functionals, *i.e.*, functions of history functions, no general solutions of the above conditions are known. However, for materials with short memory effects, constitutive functions can be reduced to ordinary functions and some general solutions for isotropic functions will be given later.

3.8 Some representation theorems

We have seen that the constitutive function must satisfy the condition (3.35) imposed by the principle of material objectivity. The general solution of this condition can be given in the following theorem. For simplicity, we shall consider purely mechanical constitutive relations for simple materials. The condition (3.35) for the stress tensor becomes

$$T = \mathcal{T}(F^t), \qquad \mathcal{T}(Q^tF^t) = Q\,\mathcal{T}(F^t)\,Q^T, \qquad \forall\, Q \in \mathcal{O}(V). \tag{3.43}$$

The extension to include all thermomechanical histories is straightforward.

Theorem. *For simple materials, the constitutive function $\mathcal{T}$ satisfies the condition of material objectivity (3.43) if and only if it can be represented by*

$$\mathcal{T}(F^t) = R\,\mathcal{T}(U^t)\,R^T, \tag{3.44}$$

where $F = RU$ is the polar decomposition of F. The restriction of $\mathcal{T}$ to the positive symmetric stretch history U^t is an arbitrary symmetric tensor-valued function.

Proof. Suppose that $\mathcal{T}(F^t)$ satisfies (3.43), then by the polar decomposition $F = RU$, we have

$$\mathcal{T}(Q^tR^tU^t) = Q\,\mathcal{T}(F^t)Q^T, \qquad \forall\, Q(t) \in \mathcal{O}(V),$$

and (3.44) follows immediately by choosing $Q = R^T$. Conversely, if (3.44) holds, then for any $Q \in \mathcal{O}(V)$, QF is nonsingular and

$$QF = QRU = (QR)U$$

is its polar decomposition. So (3.44) implies that

$$\mathcal{T}(Q^t F^t) = (QR)\,\mathcal{T}(U^t)\,(QR)^T = Q\,(R\,\mathcal{T}(U^t)\,R^T)\,Q^T = Q\,\mathcal{T}(F^t)\,Q^T.$$

Therefore (3.44) is also sufficient for (3.43). □

Constitutive functions of a simple material are required to satisfy both the condition of material objectivity (3.43) and the condition of symmetry (3.40). In order to exploit these conditions for isotropic materials, let us rewrite the constitutive function $\mathcal{T}$ as a function $\widetilde{\mathcal{T}}$ of the form,

$$T = \mathcal{T}(F^t(s)) = \widetilde{\mathcal{T}}(F^t_t(s), F(t)), \tag{3.45}$$

where $F^t_t(s) = F_t(t-s)$ denotes the history of the relative deformation gradient defined by (1.27) and by (1.28),

$$F_t(t-s) = F(t-s)\,F(t)^{-1}.$$

In terms of the function $\widetilde{\mathcal{T}}$ the conditions (3.43) and (3.40) become

$$\begin{aligned} &\widetilde{\mathcal{T}}(Q^t F^t_t Q^T, QF) = Q\,\widetilde{\mathcal{T}}(F^t_t, F)\,Q^T, && \forall\, Q \in \mathcal{O}(V), \\ &\widetilde{\mathcal{T}}(F^t_t, FG) = \widetilde{\mathcal{T}}(F^t_t, F), && \forall\, G \in \mathcal{G}. \end{aligned} \tag{3.46}$$

We can obtain general constitutive relations for a simple fluid, whose material symmetry group is the unimodular group, satisfying the conditions of material objectivity and symmetry.

Theorem. *For a simple fluid, the constitutive function $\widetilde{\mathcal{T}}$ satisfies the conditions of material objectivity and symmetry (3.46) if and only if it can be represented by*

$$\widetilde{\mathcal{T}}(F^t_t, F) = \widehat{\mathcal{T}}(U^t_t, \rho), \tag{3.47}$$

where U_t is the relative stretch tensor and ρ is the mass density in the present configuration. Moreover, $\widehat{\mathcal{T}}$ is an arbitrary isotropic symmetric tensor-valued function, i.e., it satisfies the condition,

$$\widehat{\mathcal{T}}(QU^t_t Q^T, \rho) = Q\,\widehat{\mathcal{T}}(U^t_t, \rho)\,Q^T, \tag{3.48}$$

for any orthogonal tensor Q.

Proof. Since for a simple fluid, the symmetry group $\mathcal{G} = \mathcal{U}(V)$, by taking the unimodular tensor $G = |\det F|^{\,1/3} F^{-1}$ the condition $(3.46)_2$ gives

$$\widetilde{\mathcal{T}}(F^t_t, F) = \widetilde{\mathcal{T}}(F^t_t,\, |\det F|^{\,1/3} 1).$$

Since $|\det F| = \rho_\kappa/\rho$, where ρ_κ is the mass density at the reference configuration, we can rewrite the function $\widetilde{\mathcal{T}}$ as

$$\widetilde{\mathcal{T}}(F^t_t, F) = \widehat{\mathcal{T}}(F^t_t, \rho).$$

The condition $(3.46)_1$ now becomes

$$\widetilde{\mathcal{T}}(F^t_t, F) = Q^T \widehat{\mathcal{T}}(Q^t F^t_t Q^T, \rho)\,Q.$$

With the decomposition $F_t^t = R_t^t U_t^t$, taking $Q^t(s) = R_t^t(s)^T$, hence $Q(t) = R_t^t(0) = 1$, we obtain (3.47). Similarly, choosing $Q^t(s) = \bar{Q}(t)\, R_t^t(s)^T$, hence $Q(t) = \bar{Q}(t)$, for any $\bar{Q}(t) \in \mathcal{O}(V)$, we obtain (3.48). The proof of sufficiency is straightforward. □

Similarly, we can also obtain general constitutive relations for a simple isotropic solid body which has the orthogonal group as its material symmetry group relative to some reference configuration.

Theorem. *For a simple isotropic solid body, the constitutive function $\widetilde{\mathcal{T}}$ satisfies the conditions of material objectivity and symmetry (3.46) if and only if it can be represented by*

$$\widetilde{\mathcal{T}}(F_t^t, F) = \widehat{\mathcal{T}}(U_t^t, B), \tag{3.49}$$

where U_t is the relative stretch tensor and $B = FF^T$ is the left Cauchy-Green tensor. Moreover, $\widehat{\mathcal{T}}$ is an arbitrary isotropic symmetric tensor-valued function, i.e., for any orthogonal tensor Q, it satisfies the condition,

$$\widehat{\mathcal{T}}(QU_t^t Q^T, QBQ^T) = Q\, \widehat{\mathcal{T}}(U_t^t, B)\, Q^T.$$

Proof. For isotropic solids, $\mathcal{G} = \mathcal{O}(V)$, by taking $F = RU$ and $G = R^T$, $(3.46)_2$ becomes

$$\widetilde{\mathcal{T}}(F_t^t, F) = \widetilde{\mathcal{T}}(F_t^t, RUR^T).$$

Since $V = RUR^T$ and $B = V^2$, the function $\widetilde{\mathcal{T}}$ can be rewritten as

$$\widetilde{\mathcal{T}}(F_t^t, F) = \widehat{\mathcal{T}}(F_t^t, B).$$

The rest of the proof is similar to that of the previous theorem. □

CHAPTER 4

Constitutive Equations of Fluids

4.1 Materials of grade n

We have seen that the constitutive functions for isotropic materials are isotropic functions satisfying the conditions (3.42). The main difficulty in finding general solutions for these conditions is due to the fact that they generally depend on all past values of thermomechanical histories. However, in most practical problems, memory effects are quite limited. In other words, most materials have only short memories, in the sense that we can assume that the history be approximated by a Taylor series expansion,

$$h^t(s) = h^t(0) + \frac{\partial h^t(s)}{\partial s}\Big|_{s=0} s + \frac{1}{2}\frac{\partial^2 h^t(s)}{\partial s^2}\Big|_{s=0} s^2 + \cdots \approx \sum_{k=0}^{n} \frac{1}{k!}\frac{\partial^k h^t(s)}{\partial s^k}\Big|_{s=0} s^k.$$

and consequently, the dependence of the constitutive functions on the history $h^t(s)$ reduces to the dependence on the derivatives of $h^t(s)$ up to the n-th order at the present time, $s = 0$. A material body with such an infinitesimal memory will be called a *material of grade n* relative to the history variable h.

For a simple fluid, from the representation (3.47) the constitutive relation can be written as

$$T(\boldsymbol{x}, t) = \mathcal{T}(C_t^t(\boldsymbol{x}, s), \rho(\boldsymbol{x}, t)), \tag{4.1}$$

where $C_t(\tau)$ is the relative right Cauchy-Green tensor defined by $C_t = (U_t)^2$. By defining

$$A_n = \frac{\partial^n C_t(\tau)}{\partial \tau^n}\Big|_{\tau=t}, \tag{4.2}$$

called the *Rivlin-Ericksen tensor* of order n, we have

$$C_t^t(s) \approx 1 - A_1 s + \cdots + \frac{(-1)^n}{n!} A_n s^n.$$

Therefore, from (4.1), we can write the constitutive relation for a simple fluid of grade n in the form

$$\mathcal{T} = \mathcal{T}(\rho, A_1, \cdots, A_n). \tag{4.3}$$

Furthermore, it is also required that the constitutive function must be isotropic, *i.e.*, it must satisfy the condition (3.48), which now takes the form,

$$\mathcal{T}(\rho, QA_1Q^T, \cdots, QA_nQ^T) = Q\,\mathcal{T}(\rho, A_1, \cdots, A_n)\,Q^T. \tag{4.4}$$

General solutions in the form of explicit representations for isotropic functions are known in the literature.

4.2 Isotropic functions

We shall give a more general discussion on isotropic functions in this section. Let ϕ, $\boldsymbol{h}$ and S be scalar-, vector- and tensor-valued functions defined on $\mathbb{R} \times V \times \mathcal{L}(V)$ respectively.

Definition. *We say that ϕ, $\boldsymbol{h}$, and S are scalar-, vector-, and tensor-valued isotropic functions respectively, if for any $s \in \mathbb{R}$, $\boldsymbol{v} \in V$, $A \in \mathcal{L}(V)$, they satisfy the following conditions:*

$$\begin{aligned} \phi(s, Q\boldsymbol{v}, QAQ^T) &= \phi(s, \boldsymbol{v}, A), \\ \boldsymbol{h}(s, Q\boldsymbol{v}, QAQ^T) &= Q\,\boldsymbol{h}(s, \boldsymbol{v}, A), \qquad \forall\, Q \in \mathcal{O}(V). \\ S(s, Q\boldsymbol{v}, QAQ^T) &= Q\,S(s, \boldsymbol{v}, A)\,Q^T, \end{aligned} \tag{4.5}$$

Isotropic functions are also called *isotropic invariants*. The definition can easily be extended to any number of scalar, vector and tensors variables.

Example. The following functions are
1) isotropic scalar invariants: $\boldsymbol{v} \cdot \boldsymbol{u}$, $\det A$, $\operatorname{tr}(A^m B^n)$, $A^m \boldsymbol{v} \cdot \boldsymbol{B}^n \boldsymbol{u}$;
2) isotropic vector invariants: $A^m \boldsymbol{v}$, $A^m B^n \boldsymbol{v}$;
3) isotropic tensor invariants: A^m, $A^m \boldsymbol{v} \otimes B^n \boldsymbol{v}$;

for any $\boldsymbol{u}, \boldsymbol{v} \in V$ and $A, B \in \mathcal{L}(V)$. □

Example. The vector product $\boldsymbol{v} \times \boldsymbol{u}$ is not an isotropic vector invariant, but it is a vector invariant relative to the proper orthogonal group $\mathcal{O}^+(V)$. To see this, we have for any $\boldsymbol{w} \in V$,

$$Q\boldsymbol{v} \times Q\boldsymbol{u} \cdot Q\boldsymbol{w} = (\det Q)\, \boldsymbol{v} \times \boldsymbol{u} \cdot \boldsymbol{w},$$

by the definition of the determinant. Hence we obtain

$$(Q\boldsymbol{v} \times Q\boldsymbol{u}) = (\det Q)\, Q(\boldsymbol{v} \times \boldsymbol{u}).$$

Therefore it is not a vector invariant because $\det Q = \pm 1$. □

Before giving representation theorems for isotropic functions let us recall the Cayley-Hamilton theorem, which state that a tensor $A \in \mathcal{L}(V)$ satisfies its characteristic equation,

$$A^3 - \mathrm{I}_A A^2 + \mathrm{II}_A A - \mathrm{III}_A 1 = 0, \tag{4.6}$$

where $\{\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A\}$ are called the *principal invariants* of A. They are the coefficients of the characteristic polynomial of A, *i.e.*,

$$\det(\lambda 1 - A) = \lambda^3 - \mathrm{I}_A \lambda^2 + \mathrm{II}_A \lambda - \mathrm{III}_A = 0. \tag{4.7}$$

Since eigenvalues of A are the roots of the characteristic equation (4.7), if A is symmetric and $\{a_1, a_2, a_3\}$ are three eigenvalues of A, distinct or not, then it follows that

$$\begin{aligned} \mathrm{I}_A &= a_1 + a_2 + a_3, \\ \mathrm{II}_A &= a_1 a_2 + a_2 a_3 + a_3 a_1, \\ \mathrm{III}_A &= a_1 a_2 a_3. \end{aligned} \tag{4.8}$$

It is obvious that $\mathrm{I}_A = \operatorname{tr} A$ and $\mathrm{III}_A = \det A$. Moreover, I_A, II_A and III_A are respectively a first order, a second order and a third order quantities of $|A|$. Let the space of symmetric linear transformations on V be denoted by $Sym(V) = \{A \in \mathcal{L}(V) \mid A = A^T\}$.

Theorem (Rivlin & Ericksen)**.** *Let $S : Sym(V) \to Sym(V)$, then it is an isotropic function if and only if it can be represented by*

$$S(A) = s_0 1 + s_1 A + s_2 A^2, \tag{4.9}$$

where s_0, s_1 and s_2 are arbitrary scalar functions of $(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A)$, the three principal invariants of A.

Proof. The proof of sufficiency is trivial, and we shall only prove the necessity of the above representations. We need the following lemma.

Lemma. *Let $S(A)$ be an isotropic tensor-valued function of a symmetric tensor A, then every eigenvector of A is an eigenvector of $S(A)$.*

To prove the lemma, let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a principal orthonormal basis of A. We consider a rotation Q of 180° about $\boldsymbol{e}_1$-axis, therefore,

$$Q\boldsymbol{e}_1 = \boldsymbol{e}_1, \quad Q\boldsymbol{e}_2 = -\boldsymbol{e}_2, \quad Q\boldsymbol{e}_3 = -\boldsymbol{e}_3.$$

Clearly, we have $QAQ^T = A$ and since $S(A)$is isotropic, it follows that

$$QS(A) = S(QAQ^T)Q = S(A)Q.$$

Therefore, we have

$$QS(A)\boldsymbol{e}_1 = S(A)Q\boldsymbol{e}_1 = S(A)\boldsymbol{e}_1,$$

i.e., $S(A)\boldsymbol{e}_1$ remains unchanged under Q. By our choice of Q, this can happen only if $S(A)\boldsymbol{e}_1$ is in the direction of $\boldsymbol{e}_1$. In other words, $\boldsymbol{e}_1$ is an eigenvector of $S(A)$. For other eigenvectors, we can use similar arguments. Hence the lemma is proved. □

From the lemma, if we express A as

$$A = a_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + a_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \tag{4.10}$$

then $S(A)$ can be expressed as

$$S(A) = b_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + b_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \tag{4.11}$$

where b_1, b_2, and b_3 are functions of A.

Suppose that the three eigenvalues a_1, a_2, and a_3 are distinct. We consider the following simultaneous equations for s_0, s_1, and s_2,

$$\begin{aligned} s_0 + a_1 s_1 + a_1^2 s_2 &= b_1, \\ s_0 + a_2 s_1 + a_2^2 s_2 &= b_2, \\ s_0 + a_3 s_1 + a_3^2 s_2 &= b_3. \end{aligned}$$

Since the determinant of the coefficient matrix does not vanish,

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) \neq 0,$$

we can solve for s_0, s_1, and s_2 which are functions of A of course. Then from (4.10), (4.11) can be written in the form

$$S(A) = s_0(A)1 + s_1(A)A + s_2(A)A^2. \tag{4.12}$$

Furthermore, since $S(A)$ is isotropic,

$$S(QAQ^T) = QS(A)Q^T, \qquad \forall Q \in \mathcal{O}(V),$$

it follows from (4.12) that

$$s_0(QAQ^T)1 + s_1(QAQ^T)A + s_2(QAQ^T)A^2 = s_0(A)1 + s_1(A)A + s_2(A)A^2,$$

and we conclude that s_i are all scalar isotropic functions,

$$s_i(QAQ^T) = s_i(A), \quad i = 0, 1, 2, \quad \forall\, Q \in \mathcal{O}(V).$$

Now it suffices to show that $s_i(A) = s_i(B)$ whenever A and B have the same principal invariants, or equivalently they have the same eigenvalues by (4.8). Thus let A and B be two symmetric tensors and assume that their eigenvalues are the same. Then by the spectral theorem, there exist orthonormal bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{d}_i\}$ such that

$$A = \sum_i a_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \qquad B = \sum_i a_i \boldsymbol{d}_i \otimes \boldsymbol{d}_i.$$

Let Q be the orthogonal tensor carrying the basis $\{\boldsymbol{d}_i\}$ into the basis $\{\boldsymbol{e}_i\}$, $Q\boldsymbol{d}_i = \boldsymbol{e}_i$. Then since

$$Q(\boldsymbol{d}_i \otimes \boldsymbol{d}_i)Q^T = (Q\boldsymbol{d}_i) \otimes (Q\boldsymbol{d}_i) = \boldsymbol{e}_i \otimes \boldsymbol{e}_i,$$

it follows that $QBQ^T = A$. But since s_i is isotropic,

$$s_i(A) = s_i(QBQ^T) = s_i(B),$$

which proves that $s_i = s_i(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A)$.

With similar arguments one can show that when A has exactly two distinct eigenvalues, $S(A)$ admits the representation (4.11)) with $s_2 = 0$. For the case when $A = a1$, every vector is an eigenvector, and hence by the lemma, we must have $S(A) = s_0 1$ which is a special case of the representation (4.9). The proof of the Corollary is straightforward. □

Corollary. *If $S(A)$ is an isotropic and linear function of A, then*

$$S(A) = \lambda\,(\operatorname{tr} A)1 + \mu\, A, \tag{4.13}$$

where λ and μ are independent of A.

Proof. Since $S(A)$ is isotropic and linear in A, we have, for any α,

$$S(A) = s_0(A)1 + s_1(A)A + s_2(A)A^2, \qquad S(\alpha A) = \alpha\, S(A),$$

It follows that

$$s_0(\alpha A)1 + s_1(\alpha A)\alpha A + s_2(\alpha A)\alpha^2 A^2 = \alpha\Big(s_0(A)1 + s_1(A)A + s_2(A)A^2\Big),$$

which implies that

$$\alpha s_0(A) = s_0(\alpha A), \qquad s_1(A) = s_1(\alpha A), \qquad s_2(A) = \alpha s_2(\alpha A).$$

By taking $\alpha = 0$ in the last two relations, we have

$$s_1(A) = s_1(0), \qquad s_2(A) = 0.$$

Therefore, by setting $s_1(0) = \mu$, we obtain

$$S(A) = s_0(A)1 + \mu A.$$

On the other hand, by isotropy, we have

$$s_0(A) = s_0(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A),$$

and from $\alpha s_0(A) = s_0(\alpha A)$, it follows that

$$\alpha s_0(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A) = s_0(\mathrm{I}_{\alpha A}, \mathrm{II}_{\alpha A}, \mathrm{III}_{\alpha A}).$$

Since

$$\mathrm{I}_{\alpha A} = \alpha \mathrm{I}_A, \qquad \mathrm{II}_{\alpha A} = \alpha^2 \mathrm{II}_A, \qquad \mathrm{III}_{\alpha A} = \alpha^3 \mathrm{III}_A,$$

by taking the derivative with respect to α, we obtain

$$s_0(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A) = \frac{\partial s_0}{\partial \mathrm{I}_A}\bigg|_{\alpha A} \mathrm{I}_A + 2\alpha \frac{\partial s_1}{\partial \mathrm{II}_A}\bigg|_{\alpha A} \mathrm{II}_A + 3\alpha^2 \frac{\partial s_2}{\partial \mathrm{III}_A}\bigg|_{\alpha A} \mathrm{III}_A,$$

where the partial derivatives are evaluated at αA i.e., $(\mathrm{I}_{\alpha A}, \mathrm{II}_{\alpha A}, \mathrm{III}_{\alpha A})$. Consequently, by setting $\alpha = 0$, it follows that

$$s_0(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A) = \frac{\partial s_0}{\partial \mathrm{I}_A}\bigg|_{A=0} \mathrm{I}_A = \lambda\, \mathrm{I}_A,$$

where λ is independent of A, i.e.,

$$s_0(A) = \lambda \operatorname{tr} A,$$

and the corollary is proved. □

Representations for isotropic functions of any number of vector and tensor variables have been extensively studied and the results are usually tabulated in the literature [14, 24, 19, 1]. We shall give here without proof another theorem for isotropic functions of one vector and one symmetric tensor variables.

Theorem. *Let ϕ, $\boldsymbol{h}$ and S be scalar-, vector- and symmetric tensor-valued functions respectively of a vector and a symmetric tensor variables. Then they are isotropic if and only if*

$$\begin{aligned} \phi &= f(\mathrm{I}_A, \mathrm{II}_A, \mathrm{III}_A,\ \boldsymbol{v}\cdot\boldsymbol{v},\ \boldsymbol{v}\cdot A\boldsymbol{v},\ \boldsymbol{v}\cdot A^2\boldsymbol{v}), \\ \boldsymbol{h} &= h_0\boldsymbol{v} + h_1 A\boldsymbol{v} + h_2 A^2\boldsymbol{v}, \\ S &= s_0 1 + s_1 A + s_2 A^2 + s_3 \boldsymbol{v}\otimes\boldsymbol{v} + s_4(A\boldsymbol{v}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes A\boldsymbol{v}) + s_5 A\boldsymbol{v}\otimes A\boldsymbol{v}, \end{aligned} \tag{4.14}$$

where the coefficients h_0 through s_5 are arbitrary functions of the variables indicated in the scalar function f.

4.3 Viscous fluids

From physical experiences, viscosity is a phenomenon associated with the rate of deformation – the greater the deformation rate, the greater the resistance to motion. Therefore, we shall consider a simple model for viscous materials given by the constitutive equation for the Cauchy stress, $T = \mathcal{T}(F, \dot{F})$.

From (3.35), the constitutive function $\mathcal{T}$ must satisfy the condition of material objectivity,

$$\mathcal{T}(QF, (QF)^{\cdot}) = Q\,\mathcal{T}(F, \dot{F})\,Q^T, \quad \forall\, Q \in \mathcal{O}(V), \ \ \forall\, F, \tag{4.15}$$

and from (3.40), the condition of material symmetry,

$$\mathcal{T}(FG, \dot{F}G) = \mathcal{T}(F, \dot{F}), \quad \forall\, G \in \mathcal{G}, \ \ \forall\, F. \tag{4.16}$$

For fluids, the symmetry group is the unimodular group $\mathcal{G} = \mathcal{U}(V)$. To obtain the restrictions of these conditions imposed on the constitutive function $\mathcal{T}$, we shall take $G = |\det F|^{1/3} F^{-1}$, obviously $|\det G| = 1$ so that G belongs to the symmetry group, and by (4.16), it follows that

$$\begin{aligned} \mathcal{T}(F, \dot{F}) &= \mathcal{T}(|\det F|^{1/3} FF^{-1}, |\det F|^{1/3} \dot{F}F^{-1}) \\ &= \mathcal{T}(|\det F|^{1/3} 1, |\det F|^{1/3} L) := \widehat{\mathcal{T}}(|\det F|, L). \end{aligned}$$

Therefore, for fluids, the material symmetry requires that the dependence of $\mathcal{T}$ on $(F, \dot{F})$ be reduced to the dependence on the determinant of F and the velocity gradient $L = \dot{F}F^{-1}$ as defined by the constitutive function $\widehat{\mathcal{T}}(|\det F|, L)$.

Furthermore, the function $\widehat{\mathcal{T}}$ must satisfy the condition of material objectivity (4.15) which becomes

$$\widehat{\mathcal{T}}(|\det(QF)|, (QF)^{\cdot}(QF)^{-1}) = Q\,\widehat{\mathcal{T}}(|\det F|, L)\,Q^T.$$

Simplifying the left-hand side and decomposing $L = D + W$ into symmetric and skew-symmetric parts, we obtain

$$\widehat{\mathcal{T}}(|\det F|, QDQ^T + QWQ^T + \dot{Q}Q^T) = Q\,\widehat{\mathcal{T}}(|\det F|, L)\,Q^T. \tag{4.17}$$

This relation must hold for any orthogonal tensor $Q(t)$.

Let us consider a transformation $Q^t(s) = Q(t-s)$ such that it satisfies the following differential equation and the initial condition:

$$\dot{Q}^t(s) + WQ^t(s) = 0, \quad Q^t(0) = 1.$$

It is known that the solution exists and can be expressed as $Q^t(s) = \exp(-sW)$ which is an orthogonal transformation because W is skew-symmetric[1]. Clearly it implies that

$$Q(t) = Q^t(0) = 1, \quad \dot{Q}(t) = \dot{Q}^t(0) = -W.$$

Therefore, for this choice of $Q(t)$, the above relation (4.17) reduces to

$$\widehat{\mathcal{T}}(\,|\det F|\,, L) = \widehat{\mathcal{T}}(\,|\det F|\,, D),$$

where D is the symmetric part of L. Hence, the constitutive function can not depend on the skew-symmetric part W of the velocity gradient. This, in turns, implies from the above relation (4.17) that $\widehat{\mathcal{T}}(\,|\det F|\,, D)$ is an isotropic tensor function,

$$\widehat{\mathcal{T}}(\,|\det F|\,, QDQ^T) = Q\,\widehat{\mathcal{T}}(\,|\det F|\,, D)\,Q^T, \quad \forall\, Q \in \mathcal{O}(V).$$

Moreover, from the conservation of mass, we have $|\det F| = \rho_\kappa/\rho$, where the mass density ρ_κ in the reference configuration is constant. Consequently, by replacing the dependence on $|\det F|$ with the mass density ρ, we obtain the constitutive equation,

$$T = \widetilde{\mathcal{T}}(\rho, D), \tag{4.18}$$

which is an isotropic tensor function.

Since the rate of stretch tensor D is related to the Rivlin-Ericksen tensor A_1 by $A_1 = 2D$, it is a simple fluid of grade 1. Moreover, by the representation theorem (4.9) the constitutive relation is given by

Reiner-Rivlin fluid. *The constitutive equation is given by*

$$\begin{aligned} &T = \alpha_0 1 + \alpha_1 D + \alpha_2 D^2, \\ &\alpha_i = \alpha_i(\rho, \mathrm{I}_D, \mathrm{II}_D, \mathrm{III}_D), \quad i = 0, 1, 2. \end{aligned} \tag{4.19}$$

This is the most general form of constitutive equations for simple fluids of grade 1. Even though this seems to be a general model for non-linear viscosity, it has been pointed out that this model is inadequate to describe some observed non-linear effects in real fluids (see Sect. 119 [22]). Nevertheless, by the use of (4.13), the special case, when only up to linear terms in the stretching tensor D are considered, leads to the most well-known model in fluid mechanics.

[1] Let $A \in \mathcal{L}(V)$, the exponential of A is defined as

$$\exp A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \qquad \exp A \in \mathcal{L}(V).$$

It follows that $(\exp A)^T = \exp A^T$ and if $AB = BA$ then $(\exp A)(\exp B) = \exp(A + B)$.

Navier-Stokes fluid. *The constitutive equation is given by*

$$T = (-p + \lambda \, \mathrm{tr}\, D)\, 1 + 2\mu\, D. \tag{4.20}$$

The coefficients λ and μ are called the *coefficients of viscosity*, while μ and $(\lambda + \frac{2}{3}\mu)$ are also known as the *shear* and the *bulk viscosities* respectively. The pressure p and the viscosities λ and μ are functions of ρ. A Navier-Stokes fluid is also known as a *Newtonian fluid* in fluid mechanics. It is usually assumed that

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0. \tag{4.21}$$

The non-negativeness of the shear and bulk viscosities can be proved from thermodynamic considerations. When the bulk viscosity vanishes identically, it is known as a *Stokes fluid*, a model adequate to describe some real fluids, for example water, and frequently used in numerical calculations.

Stokes fluid. *The stress tensor is given by*

$$T = -p\, 1 + 2\mu\, \hat{D}, \tag{4.22}$$

where $\hat{D}$ *is the traceless part of* D*, which in component forms is given by*

$$\hat{D}_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) - \frac{1}{3}\frac{\partial v_k}{\partial x_k}\,\delta_{ij}.$$

A Navier-Stokes fluid is governed by the system of equations consists of the conservation of mass (2.22),

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho\, v_k) = 0, \tag{4.23}$$

and the equation of motion (2.29),

$$\rho\Big(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\Big) + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_i}\Big(\lambda \frac{\partial v_k}{\partial x_k}\Big) - \frac{\partial}{\partial x_k}\Big(\mu \frac{\partial v_k}{\partial x_i}\Big) - \frac{\partial}{\partial x_k}\Big(\mu \frac{\partial v_i}{\partial x_k}\Big) = \rho\, b_i. \tag{4.24}$$

The last equation (4.24) is known as the *Navier-Stokes equation*. The pressure p and the viscosities, λ and μ, are in general functions of the density ρ. Equations (4.23) and (4.24) form a system for the fields $(\rho(\boldsymbol{x}, t),\, \boldsymbol{v}(\boldsymbol{x}, t))$. For Stokes fluids, the governing equations are obtained from above by substitution of $\lambda = -\frac{2}{3}\,\mu$.

We remark that the linear constitutive relation in (4.20) need not be regarded as an approximation of the more general constitutive relation (4.19). Any particular form of a constitutive relation (4.19) characterizes a particular class of simple fluids. Thus it is conceivable that there are some fluids which obey the constitutive equation (4.20) for *arbitrary* rate of deformation. Indeed, water and air are usually treated as Navier-Stokes fluids in most practical applications with very satisfactory results.

The simplest constitutive equation used in continuum mechanics is that of an elastic fluid, which is inviscid, *i.e.*, no viscosities. In the following constitutive relation we have included the dependence of temperature in general.

Elastic fluid. *The constitutive variables are* (ρ, θ), *therefore the we have*

$$T = -p(\rho, \theta)\, 1, \qquad \boldsymbol{q} = 0, \qquad \varepsilon = \varepsilon(\rho, \theta). \tag{4.25}$$

For an elastic fluid, from the balance equations of mass, linear momentum and energy, we have the following governing equations in component forms:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho\, v_j) = 0, \\
&\frac{\partial}{\partial t}(\rho\, v_i) + \frac{\partial}{\partial x_j}(\rho\, v_i v_j + p\, \delta_{ij}) = \rho\, b_i, \\
&\frac{\partial}{\partial t}(\rho\, \varepsilon + \frac{1}{2}\rho\, v^2) + \frac{\partial}{\partial x_j}\Big(\rho\varepsilon\, v_j + \frac{1}{2}\rho\, v^2 v_j + p\, v_j\Big) = \rho\, r.
\end{aligned} \tag{4.26}$$

The internal energy $\varepsilon = \varepsilon(\rho, \theta)$ and the pressure $p = p(\rho, \theta)$ have to be specified, for instance by the ideal gas laws, and the equations becomes a hyperbolic system for the fields $(\rho(\boldsymbol{x}, t)$, $\boldsymbol{v}(\boldsymbol{x}, t)$, $\theta(\boldsymbol{x}, t))$, usually known as the Euler equations for compressible flows of ideal gases.

4.4 Viscous heat-conducting fluids

We consider a simple fluid of grade 1 relative to mechanical histories and of grade 0 relative to thermal histories, *i.e.*,

$$\mathcal{C} = \mathcal{F}(\rho, \theta, \boldsymbol{g}, D). \tag{4.27}$$

This class of simple fluids can describe flow properties with viscosity and heat conduction. From the representation (4.14), one can immediately write down the most general constitutive equations for the stress, the heat flux and the internal energy.

Viscous heat-conducting fluid. *The constitutive equations are given by*

$$\begin{aligned}
&T = \alpha_0 1 + \alpha_1 D + \alpha_2 D^2 + \alpha_3 \boldsymbol{g} \otimes \boldsymbol{g} + \alpha_4 (D\boldsymbol{g} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes D\boldsymbol{g}) + \alpha_5 D\boldsymbol{g} \otimes D\boldsymbol{g}, \\
&\boldsymbol{q} = \beta_1 \boldsymbol{g} + \beta_2 D\boldsymbol{g} + \beta_3 D^2 \boldsymbol{g}, \\
&\varepsilon = \varepsilon(\rho, \theta, \mathrm{I}_D, \mathrm{II}_D, \mathrm{III}_D,\ \boldsymbol{g}\cdot\boldsymbol{g},\ \boldsymbol{g}\cdot D\boldsymbol{g},\ \boldsymbol{g}\cdot D^2\boldsymbol{g}),
\end{aligned} \tag{4.28}$$

where the coefficient α_i *and* β_j *as well as* ε *are scalar functions of eight variables indicated in the arguments of* ε.

The special case, when only up to linear terms in both D and $\boldsymbol{g}$ are considered, gives the following well-known model for viscosity and heat conduction in common use.

Navier-Stokes-Fourier fluids. *The constitutive equations for the stress tensor and the heat flux are given by*

$$\begin{aligned}
&T = (-p + \lambda\ \mathrm{tr}\, D)\, 1 + 2\mu\, D, \\
&\boldsymbol{q} = -\kappa\, \boldsymbol{g},
\end{aligned} \tag{4.29}$$

where the coefficients as well as the internal energy are functions of (ρ, θ).

These are the classical *Navier-Stokes* theory and the *Fourier's law* of heat conduction. The material parameters λ, μ are the viscosity coefficients and κ is called the thermal conductivity. The thermal conductivity is usually assumed to be non-negative, which can also be proved from thermodynamic considerations. The governing equations for the fields $(\rho(\boldsymbol{x},t)$, $\boldsymbol{v}(\boldsymbol{x},t)$, $\theta(\boldsymbol{x},t))$ now consist of the mass conservation (4.23), the Navier-Stokes equation (4.24) and the equation for energy (2.33) with the Navier-Stokes stress and the Fourier's law (4.29).

4.5 Incompressibility

A motion is called *incompressible* if it is volume-preserving, which can be characterized by $\det F = 1$. We call a body *incompressible material body* if it is capable of undergoing only incompressible motions.

In the formulation of constitutive relations discussed in the previous chapter, it is assumed that a material body is capable of undergoing any compressible or incompressible motions. Obviously, for incompressible bodies, some constitutive assumptions must be modified. Indeed, in order to maintain the constant volume some internal stress is needed to counter the effect of change of volume due to applied forces on the body. This is called the reaction stress and it will not do any real works in the motion. Since the rate of work in the motion due to the reaction stress N can be expressed as $(N \cdot \operatorname{grad} \boldsymbol{v})$, we shall require that

$$N \cdot \operatorname{grad} \boldsymbol{v} = N \cdot \dot{F}F^{-1} = 0, \tag{4.30}$$

for any incompressible motion. Taking the material time derivative of the equation $\det F = 1$, we obtain

$$F^{-T} \cdot \dot{F} = 1 \cdot \dot{F}F^{-1} = 0.$$

By comparison, we conclude that the reaction stress N must be proportional to the identity tensor, so let us write,

$$N = -p\,1.$$

Therefore for an incompressible body, the stress tensor can be expressed as

$$T = -p\,1 + S, \tag{4.31}$$

where p, called the *indeterminate pressure*, is a function that can not be uniquely determined by the motion, and the *extra stress* S is a constitutive quantity subject to the general requirements of material objectivity and material symmetry discussed in the previous sections.

Incompressible Navier-Stokes-Fourier fluid. *The constitutive equation is given by*

$$\begin{aligned} &T = -p\,1 + 2\mu(\theta)\,D, \qquad \operatorname{tr} D = 0, \\ &\boldsymbol{q} = -\kappa(\theta)\,\boldsymbol{g}, \end{aligned} \tag{4.32}$$

where p is an indeterminate pressure.

The condition $\operatorname{tr} D = 0$ ensures that the flow is incompressible, and hence the mass density is constant, $\rho = \rho_0$, and

$$\operatorname{div} \boldsymbol{v} = 0. \tag{4.33}$$

The Navier-Stokes equation can now be written as

$$\rho\, \dot{\boldsymbol{v}} + \operatorname{grad} p - \operatorname{div}(\mu(\theta) \operatorname{grad} \boldsymbol{v}) = \rho\, \boldsymbol{b}. \tag{4.34}$$

In an isothermal process $\theta = \theta_0$, the equation (4.34) reduces to

$$\rho\, \dot{\boldsymbol{v}} + \operatorname{grad} p - \mu\, \nabla^2 \boldsymbol{v} = \rho\, \boldsymbol{b}, \tag{4.35}$$

where $\mu = \mu(\theta)$ is a constant. The two equations (4.33) and (4.35) become the governing equations for the fields $(p(\boldsymbol{x}, t),\ \boldsymbol{v}(\boldsymbol{x}, t))$.

4.6 Viscometric flows

When a particular class of motions is concerned the generality of constitutive relations could be severely restricted resulting in a much simpler representation of constitutive equations. In this section we shall consider a general class of motions, called viscometric flows, for simple fluids. This class of motions has a considerable importance in both theoretical and experimental investigations of nonlinear viscosities.

We consider a *viscometric flow* defined by the following velocity field,

$$\boldsymbol{v}(\boldsymbol{x}, t) = u(y)\, \boldsymbol{e}_x, \tag{4.36}$$

where $\boldsymbol{x} = (x, y, z)$ and $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ is the natural basis of a coordinate system such that $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are constant unit vector fields. This class of flows includes simple shearing channel flows in a Cartesian coordinate system, as well as Poiseulle and Couette flows in a cylindrical coordinate system (r, θ, z) by regarding the r- and the z-axes as the y- and the x-axes in (4.36) respectively.

One can check easily that $|\det F| = 1$, therefore the flow is incompressible. From (1.28) the relative deformation gradient is given by

$$F_t(t-s) = F(t-s)F(t)^{-1} = 1 - s\kappa\, N,$$

where we have defined

$$\kappa = \frac{du}{dy}, \qquad N = \boldsymbol{e}_x \otimes \boldsymbol{e}_y. \tag{4.37}$$

With $C_t = F_t^T F_t$ it follows that

$$C_t(t-s) = 1 - s\, A_1 + \frac{1}{2} s^2 A_2, \tag{4.38}$$

where

$$A_1 = \kappa(N + N^T), \qquad A_2 = 2\kappa^2 N^T N \tag{4.39}$$

are the Rivlin-Ericksen tensors (see (4.2)). Therefore, from (4.1) and (4.3) we conclude that as long as viscometric flows are concerned the most general simple fluids are simple fluids of grade 2 and the constitutive relation can be expressed as

$$T = \mathcal{T}(A_1, A_2), \tag{4.40}$$

and by (4.4) the function $\mathcal{T}$ must satisfy

$$\mathcal{T}(QA_1Q^T, QA_2Q^T) = Q\,\mathcal{T}(A_1, A_2)\,Q^T, \tag{4.41}$$

for any $Q \in \mathcal{O}(V)$.

An immediately consequence follows from the condition (4.41) by choosing Q as the 180° rotation about the z-axis. From (4.39) it is easy to verify that

$$QA_1Q^T = A_1, \qquad QA_2Q^T = A_2,$$

hence (4.41) implies that

$$T_{13} = T_{31} = T_{23} = T_{32} = 0,$$

where T_{ij} denotes the components of T relative to the basis $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$. Therefore, we can write (4.40) in the following form without loss of generality,

$$T = -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_1 & \tau & 0 \\ \tau & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.42}$$

where the pressure p and the three other parameters τ, σ_1 and σ_2 are scalar functions of A_1 and A_2, or by (4.39) they are functions of κ only. We call τ the *shear stress* viscometric function, and call σ_1 and σ_2 the *normal stress* viscometric functions.

Again, by the condition (4.41), we have

$$\mathcal{T}(0,0) = Q\,\mathcal{T}(0,0)\,Q^T, \qquad \forall\, Q \in \mathcal{O}(V),$$

which implies that

$$\mathcal{T}(0,0) = \alpha\,1,$$

for some constant α. Therefore, from (4.42) it follows that

$$\alpha = -p(0),$$

and

$$\tau(0) = \sigma_1(0) = \sigma_2(0) = 0. \tag{4.43}$$

Moreover, by choosing Q as the 180° rotation about the y-axis, from (4.39) it is easy to verify that

$$QA_1Q^T = -A_1, \qquad QA_2Q^T = A_2.$$

Hence (4.41) implies that

$$\tau(-\kappa) = -\tau(\kappa), \qquad \sigma_1(-\kappa) = \sigma_1(\kappa), \qquad \sigma_2(-\kappa) = \sigma_2(\kappa). \tag{4.44}$$

In other words, $\tau(\kappa)$ is an odd function, while $\sigma_1(\kappa)$ and $\sigma_2(\kappa)$ are even functions. In summary, we have

Proposition. *The most general constitutive equation of the stress tensor for a simple fluid in a viscometric flow,* $\boldsymbol{v}(\boldsymbol{x}, t) = u(y)\, \boldsymbol{e}_x$, *is given by*

$$T = -p(\kappa)\, 1 + \tau(\kappa)\, (N + N^T) + \sigma_1(\kappa)\, NN^T + \sigma_2(\kappa)\, N^T N, \tag{4.45}$$

where

$$\kappa = \frac{du}{dy}, \qquad N = \boldsymbol{e}_x \otimes \boldsymbol{e}_y,$$

and the three viscometric functions $\tau(\kappa)$, $\sigma_1(\kappa)$ *and* $\sigma_2(\kappa)$ *satisfy the conditions* (4.43) *and* (4.44).

For *incompressible* simple fluids, the above results remain valid, except that the pressure p is not longer a material parameter and has to be determined as a field variable from the governing equations.

When the amount of shearing κ is small, the conditions (4.43) and (4.44) imply

$$\tau(\kappa) = \mu\kappa + o(k^2), \qquad \sigma_1(\kappa) = \sigma_2(\kappa) = o(\kappa^2).$$

In other words, the normal stress effects in viscometric flows are second order effects (see Sect. 23 [25, 2]). For the linear case,

$$\tau(\kappa) = \mu\kappa, \qquad \sigma_1(\kappa) = \sigma_2(\kappa) = 0,$$

the constitutive equation (4.45) reduces to (4.20) of the Navier-Stokes fluids. We shall also point out that for viscometric flows, the Reiner-Rivlin fluids are merely special cases of the most general one defined by (4.40), since its constitutive function depends only on $A_1 = 2D$ and is independent of A_2. Indeed, experimental evidences have shown that the non-linearity of the Reiner-Rivlin fluids in A_1 are inadequate to describe the observed normal stress effects (see Sect. 119 [22]).

CHAPTER 5

Constitutive Equations of Solids

5.1 Elastic materials

A simple material in general may have some memory of past deformation histories. Here, we shall consider the most important class of materials called elastic materials which has no memory of past histories at all. Some class of materials with memory effects will be briefly discussed later.

Elastic material. *The constitutive equation for the stress tensor is given by*

$$T = \mathcal{T}(F). \tag{5.1}$$

The constitutive function $\mathcal{T}$ has to satisfy the conditions of material objectivity (3.43) and material symmetry (3.40), *i.e.*,

$$\begin{aligned} &\mathcal{T}(QF) = Q\,\mathcal{T}(F)\,Q^T, && \forall\, Q \in \mathcal{O}(V), \\ &\mathcal{T}(FG) = \mathcal{T}(F), && \forall\, G \in \mathcal{G}, \end{aligned} \tag{5.2}$$

where $\mathcal{G}$ is the symmetry group of the material body. The general solution of the first condition has been obtained in (3.44), namely,

$$\mathcal{T}(F) = R\,\mathcal{T}(U)\,R^T, \tag{5.3}$$

where $F = RU$ is the polar decomposition. This representation takes a simpler form if we define

$$S = F^{-1}TF^{-T},$$

and hence by (5.3)

$$S = U^{-1}R^T\,(R\,\mathcal{T}(U)\,R^T)\,R\,U^{-T} = U^{-1}\mathcal{T}(U)\,U^{-T} = \mathcal{S}(C),$$

where we have redefined the function $\mathcal{T}$ by the new function $\mathcal{S}$ depending on the right Cauchy-Green tensor $C = U^2 = F^TF$. Therefore, the constitutive equation (5.3) which satisfies the condition of material objectivity takes the form,

$$\mathcal{T}(F) = F\,\mathcal{S}(C)\,F^T,$$

for any arbitrary function $\mathcal{S}$. By $(5.2)_2$ we have

$$FG\,\mathcal{S}(G^TF^TFG)\,G^TF^T = F\,\mathcal{S}(F^TF)\,F^T,$$

and hence the condition for material symmetry requires

$$\mathcal{S}(G^TCG) = G^{-1}\mathcal{S}(C)\,G^{-T}, \qquad \forall\, G \in \mathcal{G}.$$

Note that for any $G \in \mathcal{G}$ the transpose G^T need not belong to $\mathcal{G}$ in general, but if $\mathcal{G} \subset \mathcal{O}(V)$, then $G^T = G^{-1}$ must belong to $\mathcal{G}$ by the definition of a group. Therefore, we have

Proposition (Anisotropic elastic solid)**.** *The constitutive equation of an elastic solid with a symmetry group $\mathcal{G} \subset \mathcal{O}(V)$ is given by*

$$\mathcal{T}(F) = F\,\mathcal{S}(C)\,F^T, \qquad C = F^TF, \tag{5.4}$$

for some function $\mathcal{S}: Sym(V) \to Sym(V)$ satisfying the following condition:

$$\mathcal{S}(QCQ^T) = Q\,\mathcal{S}(C)\,Q^T, \qquad \forall\, Q \in \mathcal{G}. \tag{5.5}$$

A function satisfying this condition (5.5) is called an *anisotropic function* or an *anisotropic invariant* relative to the group $\mathcal{G}$, if $\mathcal{G}$ is a proper subgroup of $\mathcal{O}(V)$, and the body is called an anisotropic elastic solid. Explicit representations for anisotropic functions can be obtained for some symmetry groups [12, 14].

If $\mathcal{G} = \mathcal{O}(V)$ then the condition (5.5) defines an isotropic function (see (4.5)) and the body is an isotropic elastic solid, which from (3.49) the constitutive equation can also be expressed by

$$T = \mathcal{T}(B), \tag{5.6}$$

where the function $\mathcal{T}$ is an isotropic function of the left Cauchy-Green tensor $B = FF^T$.

5.2 Linear elasticity

The classical theory of linear elasticity is based on the assumption of small displacement gradient considered in Section 1.4. So let the displacement gradient H be a small quantity of order $o(1)$, and since $F = 1 + H$, we have

$$C = F^TF = 1 + H + H^T + H^TH = 1 + 2E + o(2),$$

where E is the infinitesimal strain tensor defined in (1.13). The function $\mathcal{S}$ of the equation (5.4) can now be approximated by

$$\mathcal{S}(C) = \mathcal{S}(1) + \boldsymbol{L}[E] + o(2),$$

where

$$\boldsymbol{L} = \frac{1}{2}\,\frac{\partial \mathcal{S}}{\partial C}\Big|_{C=1}$$

is a fourth order tensor which is a linear transformation of the space of symmetric tensors into itself. If we further assume that the reference configuration is a natural state, *i.e.*,

$\mathcal{T}(1) = 0$ and so is $\mathcal{S}(1) = 0$, then by neglecting the second order terms in (5.4) we obtain the linear stress-strain law,

$$T = \boldsymbol{L}[E], \tag{5.7}$$

since $\boldsymbol{L}[E]$ is of order $o(1)$ and $F = 1 + o(1)$. This linear stress-strain relation is also known as the *Hooke's law* and $\boldsymbol{L}$ is called the *elasticity tensor*. By definition the elasticity tensor has the following symmetry properties in terms of components:

$$L_{ijkl} = L_{jikl} = L_{ijlk}, \tag{5.8}$$

and an additional symmetry,

$$L_{ijkl} = L_{klij}, \tag{5.9}$$

if the material is *hyperelastic*, *i.e.*, there exists a function $\psi(F)$, called a stored energy function, such that

$$T = \frac{\partial \psi}{\partial E}. \tag{5.10}$$

The existence of a stored energy function will be proved later from thermodynamic considerations.

Moreover, the conditions of material objectivity and material symmetry (5.2) imply that

$$\mathcal{T}(QFQ^T) = Q\,\mathcal{T}(F)\,Q^T, \qquad \forall\, G \in \mathcal{Q} \subset \mathcal{O}(V).$$

Since

$$T(F) = \boldsymbol{L}[E(F)], \qquad E(F) = \frac{1}{2}(H + H^T) = \frac{1}{2}(F + F^T) - 1,$$

it follows immediately that

$$\boldsymbol{L}[QEQ^T] = Q\,\boldsymbol{L}[E]\,Q^T, \qquad \forall\, Q \in \mathcal{G} \subset \mathcal{O}(V). \tag{5.11}$$

This relation can be written in component forms,

$$L_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}\,L_{mnpq}.$$

Finally, by (1.16) and the symmetry condition (5.8), the equation of motion (2.29) for linear elasticity in component forms is given by

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\Big(L_{ijkl}\frac{\partial u_k}{\partial x_l}\Big) + \rho\, b_i. \tag{5.12}$$

This is the governing equation for the displacement vector $\boldsymbol{u}$.

Remark. We should point out that the linear law (5.7) does not satisfy the condition of material objectivity $(5.2)_1$ for arbitrary orthogonal tensor Q. Therefore, unlike the Navier-Stokes theory, which does satisfy the condition of material objectivity, the theory of linear elasticity is meaningless for large deformations.

Indeed, if we choose $F = 1$, which is a natural state by assumption, then the condition $(5.2)_1$ implies that

$$T(Q) = QT(1)Q^T = 0,$$

for any orthogonal tensor Q. On the other hand, since

$$T(F) = \boldsymbol{L}[E(F)], \qquad E(F) = \frac{1}{2}(H + H^T) = \frac{1}{2}(F + F^T) - 1,$$

if we choose Q as a rotation about z-axis,

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$E(Q) = \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and for $\theta \neq 0$

$$T(Q) = \boldsymbol{L}[E(Q)] \neq 0.$$

Hence, the condition of material objectivity is not satisfied in general. Nevertheless, one can show that it is approximately satisfied when both the displacement and the rotation are small (in this case, $\cos\theta \approx 1$). □

5.3 Isotropic elastic solids

For an isotropic solid with no memory of past histories, from the constitutive relation (5.6) and the representation theorem (4.9), we have

Isotropic elastic solid. *The constitutive equation can be written as*

$$T = t_0 1 + t_1 B + t_2 B^2, \tag{5.13}$$

where t_i for $i = 0, 1, 2$ are functions of $(\mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B)$, the principal invariants of B.

This is the general constitutive equation for isotropic *finite elasticity*, the theory of elastic solids for *finite* deformations. It can also be obtained as a special case from the constitutive equation (5.4) with $\mathcal{S}$ being an isotropic function of C by (5.5), since the symmetry group $\mathcal{G}$ is the orthogonal group $\mathcal{O}(V)$.

For small deformations, we have

$$B = FF^T = 1 + 2E + o(2),$$

and hence the linear approximation of $\mathcal{T}(B)$ at $B = 1$ can be written as

$$\mathcal{T}(B) = 2\,\partial_B \mathcal{T}(1)[E] + o(2),$$

where the reference configuration is assumed to be a natural state as before. By explicitly carrying out the gradient $\partial_B \mathcal{T}$ from (5.13) and neglecting the second order terms, we obtain

$$T = \lambda\,(\operatorname{tr} E)1 + 2\mu\, E, \tag{5.14}$$

where λ and μ are called the *Lamé elastic moduli* and they are related to the material parameters t_0, t_1, and t_2 of (5.13) by

$$\begin{aligned} &\lambda = 2\left(\frac{\partial t}{\partial \mathrm{I}_B} + 2\frac{\partial t}{\partial \mathrm{II}_B} + \frac{\partial t}{\partial \mathrm{III}_B}\right)\Big|_{(3,3,1)} \qquad t = t_0 + t_1 + t_2,\\ &\mu = t_1(3,3,1) + 2\,t_2(3,3,1). \end{aligned} \tag{5.15}$$

The equation (5.14) is the constitutive relation of the classical theory of isotropic *linear elasticity*. It is a special case of (5.7) with the elasticity tensor given by

$$L_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{5.16}$$

For an isotropic linear elastic body, the equation of motion becomes

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_i}\Big(\lambda\frac{\partial u_k}{\partial x_k}\Big) + \frac{\partial}{\partial x_k}\Big(\mu\big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\big)\Big) + \rho\, b_i. \tag{5.17}$$

If the body is homogeneous, then λ and μ are constants, and the governing equation becomes

$$\rho\,\ddot{\boldsymbol{u}} = (\lambda + \mu)\ \mathrm{grad}(\operatorname{div}\boldsymbol{u}) + \mu\,\nabla^2\boldsymbol{u} + \rho\,\boldsymbol{b}. \tag{5.18}$$

The linear stress-strain relation (5.14) can also be derived directly from (5.7) for the symmetry group $\mathcal{G} = \mathcal{O}(V)$. Indeed, if we define

$$S(E) = \boldsymbol{L}[E],$$

then by (5.11) the symmetric tensor-valued function S is an isotropic function and it is linear in the symmetric tensor variable E. Therefore, from the representation theorem (4.13), we can represent $S(E)$ as an expression linear in E in the following form,

$$S(E) = \lambda\,(\operatorname{tr} E)1 + 2\mu\, E.$$

5.4 Incompressible elastic solids

Incompressible elastic bodies can be similarly formulated. It has been shown that the reaction stress for incompressibility is a hydrostatic pressure and the stress tensor takes the form (4.31).

Incompressible elastic material. *The constitutive equation for the stress tensor is given by*

$$T = -p1 + \mathcal{T}(F), \qquad \det F = 1, \tag{5.19}$$

where p is the indeterminate hydrostatic pressure.

For incompressible isotropic elastic solids, (5.19) reduces to $T = -p1 + \mathcal{T}(B)$, where the function $\mathcal{T}$ is an isotropic function. Hence from the representation (5.13) the constitutive equation can be written as

$$T = -p\,1 + t_1 B + t_2 B^2,$$

or equivalently, by the use of Cayley-Hamilton theorem,

$$T = -p\,1 + s_1 B + s_{-1} B^{-1}, \tag{5.20}$$

where B is required to be unimodular, *i.e.*, $\det B = 1$ and in general the parameters s_1 and s_{-1} are functions of $(\mathrm{I}_B, \mathrm{II}_B)$, since $\mathrm{III}_B = 1$ for incompressibility. Is is usually assumed that

$$s_1 > 0, \quad s_{-1} \leq 0. \tag{5.21}$$

This assumption is known as (empirical) *E-inequalities* in elasticity (see Sect. 55 [22] for more discussions on experimental data for rubber).

Two special cases are of practical interest for finite elasticity, namely, the simple models for which the parameters s_1 and s_{-1} are constants.

Neo-Hookean material. *The stress tensor takes the form,*

$$T = -p\,1 + s_1 B, \qquad \det B = 1, \tag{5.22}$$

where s_1 is a positive constant parameter.

Mooney-Rivlin material. *The stress tensor takes the form,*

$$T = -p\,1 + s_1 B + s_{-1} B^{-1}, \qquad \det B = 1, \tag{5.23}$$

where $s_1 > 0$ and $s_{-1} \leq 0$ are constants.

These incompressible material models are often adopted for rubber-like materials. Neo-Hookean materials are also predicted by the kinetic theory of rubber from molecular calculations in the first approximation [20]. It provides a reasonable theory of natural rubber for modest strains.

5.5 Thermoelastic materials

Heat conduction in elastic materials is usually taken into account by including the temperature θ and the temperature gradient $\boldsymbol{g}$ as constitutive variable.

Thermoelastic material. *The constitutive equations are given in the form,*

$$\mathcal{C} = \mathcal{F}(F, \theta, \boldsymbol{g}). \tag{5.24}$$

Beside the requirements of material objectivity and material symmetry, for thermoelastic materials, thermodynamic considerations are essential to obtain a great simplification to constitutive equations. We shall state here the main results for further discussions and leave the proof in the next chapter[1].

Proposition. *For thermoelastic materials, there exists a function $\psi = \psi(F, \theta)$, such that the stress tensor T and the internal energy ε are given by*

$$T = \rho \frac{\partial \psi}{\partial F} F^T, \qquad \varepsilon = \psi - \theta \frac{\partial \psi}{\partial \theta}, \tag{5.25}$$

and the heat flux $\boldsymbol{q}$ satisfies the condition,

$$\boldsymbol{q} \cdot \boldsymbol{g} \leq 0. \tag{5.26}$$

Consequently, the stress T and the internal energy ε are independent of the temperature gradient $\boldsymbol{g}$ and they are completely determined by a scalar function $\psi(F, \theta)$, called the *free energy function*. The relation $(5.25)_1$ also states that the material body is *hyperelastic*.

For isotropic thermoelastic solid bodies, by (5.6) and the representation theorem (4.14) the most general constitutive functions can now be written down immediately.

Isotropic thermoelastic solid. *The constitutive equations for the stress tensor and the heat flux vector are given by*

$$\begin{aligned} T &= t_0 1 + t_1 B + t_2 B^2, \\ \boldsymbol{q} &= k_1 \boldsymbol{g} + k_2 B \boldsymbol{g} + k_3 B^2 \boldsymbol{g}, \end{aligned} \tag{5.27}$$

The internal energy ε and the coefficients t_i are scalar functions of $(\theta, \mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B)$, while the coefficients k_i are functions of $(\theta, \mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B,\ \boldsymbol{g} \cdot \boldsymbol{g},\ \boldsymbol{g} \cdot B\boldsymbol{g},\ \boldsymbol{g} \cdot B^2\boldsymbol{g})$.

Further restrictions on the coefficient functions can be obtained from the relations (5.25).

It is more convenient to introduce a function η called the *entropy density*, which will be considered in more details later, by $\varepsilon = \theta\eta + \psi$. The relations (5.25) becomes,

$$T = \rho \frac{\partial \psi}{\partial F} F^T, \qquad \eta = -\frac{\partial \psi}{\partial \theta}. \tag{5.28}$$

[1]It will be proved for isotropic elastic materials and elastic materials of Fourier type based on the general entropy inequality. However, it can also be proved for elastic materials in general based on the more restricted Clausius-Duhem inequality.

From these relations we can obtain

$$\dot{\varepsilon} = \theta\dot{\eta} + \frac{1}{\rho}TF^{-T}\cdot\dot{F},$$

and hence the energy equation (2.33) for thermoelastic materials can be written as

$$\rho\theta\,\dot{\eta} + \operatorname{div}\boldsymbol{q} = \rho\, r. \tag{5.29}$$

5.6 Linear thermoelasticity

For the classical linear theory, the displacement gradient H and the temperature increment $\tilde{\theta} = \theta - \theta_0$ are assumed to be small quantities, where θ_0 is the reference temperature of the body. On the other hand, similar to (3.44) the principle of material objectivity requires that the dependence of the free energy function on the deformation gradient F must reduces to the dependence of the right stretch tensor U. Since $U = 1 + E + o(2)$ by (1.12), in the linear theory we have $\psi = \psi(E,\theta)$. Therefore let us express the function ψ up to the second order terms in E and $\tilde{\theta}$ in the following form,

$$\psi = \psi_0 - \eta_0\tilde{\theta} + M_{ij}E_{ij} - \frac{1}{2}\frac{c_v}{\theta_0}\tilde{\theta}^2 - \frac{1}{\rho_0}P_{ij}E_{ij}\tilde{\theta} + \frac{1}{2}L_{ijkl}E_{ij}E_{kl}. \tag{5.30}$$

The relations (5.28) now take the form,

$$T = \rho_0\frac{\partial\psi}{\partial E}, \qquad \eta = -\frac{\partial\psi}{\partial\tilde{\theta}},$$

and if the reference state is assumed to be a natural state, then M_{ij} in (5.30) must vanish and we obtain

$$\begin{aligned} T_{ij} &= L_{ijkl}E_{kl} - P_{ij}\tilde{\theta}, \\ \eta &= \eta_0 + \frac{c_v}{\theta_0}\,\tilde{\theta} + \frac{1}{\rho_0}P_{ij}E_{ij}. \end{aligned} \tag{5.31}$$

The fourth order tensor $\boldsymbol{L}$ is the elasticity tensor (see (5.7)) and c_v is called the *specific heat* because from $(5.25)_2$ it follows that

$$c_v = \frac{\partial\varepsilon}{\partial\theta}.$$

For the linear theory, we shall assume in addition that the Fourier's law of heat conduction holds, so that the linear expression for the heat flux is given by

$$q_i = -K_{ij}\,g_j, \tag{5.32}$$

where K is the thermal conductivity tensor.

Therefore, we can summarize the constitutive equations of linear thermoelasticity for anisotropic materials in the following:

$$\begin{aligned} T &= \mathbf{L}[E] - P\,\tilde{\theta}, \\ \boldsymbol{q} &= -K\,\boldsymbol{g}, \\ \eta &= \eta_0 + \frac{c_v}{\theta_0}\,\tilde{\theta} + \frac{P}{\rho_0}\cdot E. \end{aligned} \tag{5.33}$$

Moreover, the coefficients satisfy the following conditions:

$$\begin{aligned} &L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}, \quad P_{ij} = P_{ji}, \\ &K \text{ is positive semi-definite}, \\ &c_v > 0. \end{aligned} \tag{5.34}$$

The first two conditions follow from the definition in (5.30) and the condition for the thermal conductivity tensor follows from (5.26). The last inequality is a consequence of thermal stability which will be discussed later.

If the material is isotropic, then we have, for any orthogonal tensor Q,

$$\begin{aligned} &T(QFQ^T, \theta) = Q\, T(F, \theta)\, Q^T, \\ &\boldsymbol{q}(QFQ^T, \theta, Q\boldsymbol{g}) = Q\, \boldsymbol{q}(F, \theta, \boldsymbol{g}). \end{aligned}$$

Since

$$E = \frac{1}{2}(H + H^T) = \frac{1}{2}(F + F^T) - 1,$$

from (5.33) it follows immediately that

$$\begin{aligned} &\mathbf{L}[QEQ^T] = Q\, \mathbf{L}[E]\, Q^T, \\ &P = QPQ^T, \\ &KQ = QK. \end{aligned}$$

Therefore, we conclude that in addition to the relation (5.16) we also have

$$P_{ij} = \alpha\, \delta_{ij}, \qquad K_{ij} = \kappa\, \delta_{ij}.$$

In summary, the constitutive equations of linear thermoelasticity for isotropic materials are given in the following:

$$\begin{aligned} &T = \lambda \operatorname{tr} E\, 1 + 2\mu\, E - \alpha\, \tilde{\theta}\, 1, \\ &\boldsymbol{q} = -\kappa\, \boldsymbol{g}, \\ &\eta = \eta_0 + \frac{c_v}{\theta_0}\, \tilde{\theta} + \frac{\alpha}{\rho_0} \operatorname{tr} E. \end{aligned} \tag{5.35}$$

The field equations for thermoelasticity consist of the momentum equation and the energy equation (or the equivalent equation (5.29)) for the displacement $\boldsymbol{u}(\boldsymbol{x}, t)$ and the temperature $\theta(\boldsymbol{x}, t)$. In component forms, from (1.16) and the constitutive equations (5.33), we have the following field equations for anisotropic thermoelastic materials:

$$\begin{aligned} &\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\Big(L_{ijkl} \frac{\partial u_k}{\partial x_l}\Big) + \frac{\partial}{\partial x_j}\Big(P_{ij}(\theta - \theta_0)\Big) = \rho_0 b_i, \\ &\rho_0 c_v \frac{\partial \theta}{\partial t} + \theta_0 P_{ij} \frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial x_j}\Big) - \frac{\partial}{\partial x_i}\Big(K_{ij} \frac{\partial \theta}{\partial x_j}\Big) = \rho_0\, r. \end{aligned} \tag{5.36}$$

In particular, if the body is homogeneous, *i.e.*, the material coefficients are independent of $\boldsymbol{X}$ (in the linear theory, it is the same as $\boldsymbol{x}$), then the field equations (5.36) become

$$\begin{aligned} &\rho_0 \frac{\partial^2 u_i}{\partial t^2} - L_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + P_{ij} \frac{\partial \theta}{\partial x_j} = \rho_0 b_i, \\ &\rho_0 c_v \frac{\partial \theta}{\partial t} + \theta_0 P_{ij} \frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial x_j}\Big) - K_{ij} \frac{\partial^2 \theta}{\partial x_i \partial x_j} = \rho_0\, r. \end{aligned} \tag{5.37}$$

For isotropic thermoelastic materials, from (5.35) the field equations become

$$\begin{aligned} &\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_i}\Big(\lambda \frac{\partial u_k}{\partial x_k}\Big) - \frac{\partial}{\partial x_j}\Big(\mu\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)\Big) + \frac{\partial}{\partial x_i}\Big(\alpha(\theta - \theta_0)\Big) = \rho_0 b_i, \\ &\rho_0 c_v \frac{\partial \theta}{\partial t} + \alpha\theta_0 \frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial x_i}\Big) - \frac{\partial}{\partial x_i}\Big(\kappa \frac{\partial \theta}{\partial x_i}\Big) = \rho_0\, r. \end{aligned} \tag{5.38}$$

If the body is homogeneous then the field equations become

$$\begin{aligned} &\rho_0 \frac{\partial^2 u_i}{\partial t^2} - (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} - \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \alpha \frac{\partial \theta}{\partial x_i} = \rho_0 b_i, \\ &\rho_0 c_v \frac{\partial \theta}{\partial t} + \alpha\theta_0 \frac{\partial}{\partial t}\Big(\frac{\partial u_k}{\partial x_k}\Big) - \kappa \frac{\partial^2 \theta}{\partial x_k \partial x_k} = \rho_0\, r. \end{aligned} \tag{5.39}$$

5.7 Mooney-Rivlin thermoelastic materials

To obtain a constitutive equation for moderate strains of an incompressible isotropic thermoelastic solid, we shall start with the free energy function $\psi = \psi(\theta, \mathrm{I}_B, \mathrm{II}_B)$, and from (5.25) that the Cauchy stress is given by

$$T = -p\, I + 2\rho \frac{\partial \psi}{\partial B} B, \qquad \det B = 1. \tag{5.40}$$

By Taylor series expansion, we can write

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3) + o(\delta_1^2, \delta_2^2, \delta_1\delta_2), \tag{5.41}$$

where $\psi_k = \psi_k(\theta)$ for $k = 0, 1, 2$, and $o(\delta_1^2, \delta_2^2, \delta_1\delta_2)$ stands for the second order terms in $\mathrm{I}_B - 3 = o(\delta_1)$ and $\mathrm{II}_B - 3 = o(\delta_2)$, say, respectively of the order of $\delta_1 < 1$ and $\delta_2 < 1$.

Some useful relations

Since $F = I + H$, we have

$$B = FF^T = I + A, \quad \text{where} \quad A = H + H^T + HH^T.$$

Therefore, if we assume that the displacement gradient H is of the order of $\delta < 1$, write, $H = o(\delta)$, it follows that A is of the same order, $A = o(\delta)$.

Note that for $B = I + A$, one can show that

$$\begin{aligned} \mathrm{I}_B &= 3 + \mathrm{I}_A, \\ \mathrm{II}_B &= 3 + 2\,\mathrm{I}_A + \mathrm{II}_A, \\ \mathrm{III}_B &= 1 + \mathrm{I}_A + \mathrm{II}_A + \mathrm{III}_A. \end{aligned}$$

In particular, for $\det B = \mathrm{III}_B = 1$, we have

$$\mathrm{I}_A = -\mathrm{II}_A - \mathrm{III}_A.$$

Therefore, since $\mathrm{II}_A = o(\delta^2)$, of the order δ^2, so is $\mathrm{I}_A = o(\delta^2)$, and we have

$$\begin{aligned} \mathrm{I}_B - 3 &= \mathrm{I}_A = o(\delta_1) = o(\delta^2), \\ \mathrm{II}_B - 3 &= 2\,\mathrm{I}_A + \mathrm{II}_A = o(\delta_2) = o(\delta^2), \end{aligned}$$

which implies $o(\delta_1^2, \delta_2^2, \delta_1\delta_2) = o(\delta^4)$. Consequently, from (5.41) we have

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3) + o(\delta^4).$$

In other words, the free energy function,

$$\psi(\theta, \mathrm{I}_B, \mathrm{II}_B) = \psi_0 + \psi_1(\mathrm{I}_B - 3) + \psi_2(\mathrm{II}_B - 3), \tag{5.42}$$

is a general representation with an error in the fourth order of the displacement gradient.

Mooney-Rivlin material

From (5.40), we obtain

$$T + p\,I = s_1 B + s_2(B^2 - \mathrm{I}_B B), \tag{5.43}$$

where

$$s_1(\theta) = 2\rho\psi_1(\theta), \qquad s_2(\theta) = -2\rho\psi_2(\theta),$$

and we have used the relations

$$\frac{\partial\,\mathrm{I}_B}{\partial B} = I, \qquad \frac{\partial\,\mathrm{II}_B}{\partial B} = \mathrm{I}_B I - B.$$

Note that the constitutive equation (5.43) can be written as

$$T = -p\,I + t_1 B + t_2 B^2,$$

with

$$t_1 = s_1 - s_2 \mathrm{I}_B, \qquad t_2 = -s_2.$$

Consequently, t_1 is not a function of θ only. However, by the use of the Cayley-Hamilton theorem, with $\mathrm{III}_B = 1$,

$$B^2 - \mathrm{I}_B B = B^{-1} - \mathrm{II}_B I,$$

it becomes

$$T = -p\,I + s_1 B + s_2 B^{-1},$$

in which the term $\mathrm{II}_B I$ is absorbed into the indeterminate pressure $p\,I$. In this case, the material parameters s_1 and s_2 are functions of the temperature only. Therefore we conclude that

Mooney-Rivlin material. *The constitutive equation*

$$T = -p\,I + s_1 B + s_2 B^{-1}, \tag{5.44}$$

with $s_1(\theta)$ and $s_2(\theta)$, is general enough to account for incompressible thermoelastic isotropic solids at moderate strain with an error in the fourth order of the displacement gradient in free energy.

Isothermal uniaxial experiment

In order to examine some possible restrictions on the values of the material parameters s_1 and s_2, we shall consider a simple isothermal uniaxial experiment: For the uniaxial deformation $\lambda > 0$,

$$x = \lambda\, X, \quad y = \frac{1}{\sqrt{\lambda}}\, Y, \quad z = \frac{1}{\sqrt{\lambda}}\, Z,$$

of a cylinder with cross-section area A, and length $0 \leq X \leq L$ in the reference state κ, we have

$$B = \lambda^2\, \boldsymbol{e}_x \otimes \boldsymbol{e}_x + \frac{1}{\lambda}\,(\boldsymbol{e}_y \otimes \boldsymbol{e}_y + \boldsymbol{e}_z \otimes \boldsymbol{e}_z),$$

$$\mathrm{I}_B = \lambda^2 + \frac{2}{\lambda}, \qquad \mathrm{II}_B = \frac{1}{\lambda^2} + 2\lambda,$$

and the free energy (5.41) in this case becomes

$$\psi(\lambda) = \psi_0 + \psi_1\Big(\lambda^2 + \frac{2}{\lambda} - 3\Big) + \psi_2\Big(\frac{1}{\lambda^2} + 2\lambda - 3\Big). \tag{5.45}$$

From thermodynamic considerations, it can be proved that in this experiment (see Sect. 6.6)

$$\frac{d}{dt}\int_V \rho\psi\, dv - \int_{\partial V} \dot{\boldsymbol{x}} \cdot T\boldsymbol{n}\, da \leq 0,$$

which can be rewritten in the reference state as

$$\frac{d}{dt}\int_{V_\kappa} \rho\psi\, dv - \int_{\partial V_\kappa} \dot{\boldsymbol{x}} \cdot T_\kappa \boldsymbol{n}_\kappa\, da \leq 0,$$

where T_κ is the Piola Kirchhoff stress on the reference boundary and in this case, we have the following boundary conditions: $T_\kappa = 0$ on the lateral boundary of the cylinder and is constant on the top and the bottom, while

$$\dot{\boldsymbol{x}}|_{X=0} = 0, \qquad \dot{\boldsymbol{x}}|_{X=L} = \dot{\lambda} L\, \boldsymbol{e}_x.$$

Therefore, it follows after integration that

$$\frac{d}{dt}\Big(\rho\psi - T_{\kappa\langle xx\rangle}\lambda\Big)AL \leq 0.$$

If we call $\Psi(\lambda) = \rho\psi(\lambda) - T_{\kappa\langle xx\rangle}\lambda$ the *available energy*, as the sum of the free energy and the potential energy, then we have

$$\frac{d\Psi}{dt} \leq 0.$$

This allows us to establish a stability criterion. If we assume the equilibrium state at the deformation λ is a stable state, then any small perturbation from this state will eventually return to this state as time tends to infinity. Suppose that such a perturbation is represented by a dynamic deformation process $\lambda(t)$ under the same boundary conditions, it follows that

$$\lim_{t\to\infty} \lambda(t) = \lambda,$$

and since $\Psi(\lambda(t))$ is a decreasing function of time, it implies that

$$\Psi(\lambda(t)) \geq \Psi(\lambda) \quad \forall\, t > 0.$$

In other words, the available energy $\Psi(\lambda(t))$ attains its minimum at λ. Therefore, we arrive at the following criterion:

Thermodynamic stability criterion. *A state at the uniaxial deformation λ is stable (as $t \to \infty$) if the value of the available energy function $\Psi(\lambda)$ attains its minimum at this state.*

The minimization of $\Psi(\lambda)$ at given λ requires that

$$\frac{d\Psi(\lambda)}{d\lambda} = 0, \quad \text{and} \quad \frac{d^2\Psi(\lambda)}{d\lambda^2} \geq 0.$$

From the expression (5.45), the first condition

$$\frac{d\Psi(\lambda)}{d\lambda} = 2\rho\psi_1\Big(\lambda - \frac{1}{\lambda^2}\Big) - 2\rho\psi_2\Big(\frac{1}{\lambda^3} - 1\Big) - T_{\kappa\langle xx\rangle} = 0,$$

implies that

$$T_{\kappa\langle xx\rangle} = s_1\Big(\lambda - \frac{1}{\lambda^2}\Big) + s_2\Big(\frac{1}{\lambda^3} - 1\Big).$$

This is the axial stress per unit original area of the cylinder in the experiment, which can also be obtained directly from the constitutive equation (5.44) for the uniaxial deformation λ.

The second condition, on the other hand, is more interesting, we have

$$\frac{d^2\Psi(\lambda)}{d\lambda^2} = 2\rho\psi_1\Big(1 + \frac{2}{\lambda^3}\Big) + 2\rho\psi_2\,\frac{3}{\lambda^4} \geq 0.$$

Since $\psi_2/\psi_1 = -s_2/s_1$ by assuming $s_1 > 0$, it follows that

$$\frac{s_2}{s_1} \leq \frac{1}{3}\,(2\lambda + \lambda^4). \tag{5.46}$$

Obviously, this is a restriction for possible values of the material parameters s_1 and s_2. In particular, for $\lambda = 1$, it gives $s_1 \geq s_2$, which, in turns, implies that the requirement is always satisfied for uniaxial extension, $\lambda > 1$. On the other hand, for uniaxial contraction as $\lambda \to 0$, it requires $s_2 \leq 0$. Therefore, the assumption (5.21),

$$s_1 > 0, \quad s_2 \leq 0,$$

satisfies the condition (5.46) for any deformation $\lambda > 0$. This is known as the E-inequality (5.21) strongly supported by experimental evidence for rubber (see Sect. 53 of [22]).

However, for most materials, it is almost impossible to reach the limit at $\lambda \to 0$ in uniaxial contraction without structure failure, and since the Mooney-Rivlin material is a good material model for thermoelastic solid with moderate strain, say, $\lambda \in (\lambda_L, \lambda_U)$, for some $\lambda_U > 1$ and $1 > \lambda_L > 0$, the condition (5.46) requires that

$$s_2 \leq \gamma\, s_1, \quad \text{for} \quad \gamma = \frac{1}{3}\,(2\lambda_L + \lambda_L^4) < 1. \tag{5.47}$$

In other words, for a material body which has a finite lower limit in uniaxial contraction, it is *not necessary* to require that the material parameter s_2 be negative as proposed by the E-inequality.

On the other hand, for a Mooney-Rivlin material model with $s_1 > s_2 = \gamma s_1 > 0$, the thermodynamic stability is no longer guaranteed for $\lambda < \lambda_L$. Experimentally, it could possibly mean failure of the material body under contraction beyond the lower limit.

5.8 Rigid heat conductors

Like incompressible material bodies, rigid bodies are also considered as bodies with internal constraints. In this case, motions of the body is restricted to rigid transformations only,

$$\boldsymbol{x}(\boldsymbol{X}, t) = R(t)(\boldsymbol{X} - \boldsymbol{X}_0) + \boldsymbol{d}(t),$$

where $R(t) \in \mathcal{O}(V)$ and $\boldsymbol{X}_0, \boldsymbol{d}(t) \in \mathcal{E}$, hence the deformation gradient $F(\boldsymbol{X}, t) = R(t)$ is an orthogonal transformation. To maintain rigid motions of the body, a reaction stress N which does not work is needed, i.e., the stress tensor

$$T(\boldsymbol{X}, t) = N(\boldsymbol{X}, t) + S(\boldsymbol{X}, t),$$

where S is the extra stress which is a constitutive quantity subject to general constitutive requirements, and $N \cdot D = 0$ for any rate of strain tensor D of the motion.

Since for rigid motions, F is orthogonal, by taking the time derivative of $FF^T = I$, we have

$$\dot{F}F^T + F\dot{F}^T = L + L^T = 0,$$

where we have used $F^T = F^{-1}$ and $L = \dot{F}F^{-1}$. Therefore, the rate of strain $D = 0$ identically in the motion, and the condition $N \cdot D = 0$ implies that the reaction stress is arbitrary. Consequently, the total stress T is completely indeterminable by the deformations and can only be determined from the equation of motion and boundary conditions of the body.

On the other hand, for the heat flux and the internal energy, they remain to be determined from constitutive equations,

$$\boldsymbol{q} = \boldsymbol{q}(F, \theta, \boldsymbol{g}), \qquad \varepsilon = \varepsilon(F, \theta, \boldsymbol{g}),$$

subject to the restrictions that the deformation gradient F must be orthogonal. They must satisfy the condition of material objectivity, in particular, for the heat flux,

$$\boldsymbol{q}(QF, \theta, Q\boldsymbol{g}) = Q\boldsymbol{q}(F, \theta, \boldsymbol{g}), \qquad \forall\, Q \in \mathcal{O}(V), \tag{5.48}$$

and the condition of material symmetry,

$$\boldsymbol{q}(FG,\theta,\boldsymbol{g}) = \boldsymbol{q}(F,\theta,\boldsymbol{g}), \qquad \forall\, G \in \mathcal{G}_\kappa, \tag{5.49}$$

where $\mathcal{G}_\kappa$ is the material symmetry group of the body and under the restriction that only orthogonal deformation gradient F is allowed.

As consequences of these requirements, from (5.48), by taking $Q = F^T \in \mathcal{O}(V)$, we have

$$\boldsymbol{q}(I,\theta,F^T\boldsymbol{g}) = F^T\boldsymbol{q}(F,\theta,\boldsymbol{g}). \tag{5.50}$$

Furthermore, for rigid solid bodies, we have $\mathcal{G}_\kappa \subseteq \mathcal{O}(V)$ and from (5.49), by taking $F = G^T$ for $G \in \mathcal{G}_\kappa$, we have

$$\boldsymbol{q}(I,\theta,\boldsymbol{g}) = \boldsymbol{q}(G^T,\theta,\boldsymbol{g}) = G^T\boldsymbol{q}(I,\theta,G\boldsymbol{g}), \tag{5.51}$$

in the last passage (5.50) is used. If we define

$$\hat{\boldsymbol{q}}(\theta,\boldsymbol{g}) = \boldsymbol{q}(I,\theta,\boldsymbol{g}),$$

from (5.50), the constitutive equation for the heat flux is given by

$$\boldsymbol{q} = \boldsymbol{q}(F,\theta,\boldsymbol{g}) = F\hat{\boldsymbol{q}}(\theta,F^T\boldsymbol{g}) \qquad \forall\, F \in \mathcal{O}(V), \tag{5.52}$$

and the relation (5.51) becomes

$$\hat{\boldsymbol{q}}(\theta,G\boldsymbol{g}) = G\hat{\boldsymbol{q}}(\theta,\boldsymbol{g}), \qquad \forall\, G \in \mathcal{G}_\kappa \subseteq \mathcal{O}(V).$$

Similarly, for the internal energy, we have

$$\varepsilon = \varepsilon(F,\theta,\boldsymbol{g}) = \hat{\varepsilon}(\theta,F^T\boldsymbol{g}) \qquad \forall\, F \in \mathcal{O}(V), \tag{5.53}$$

and

$$\hat{\varepsilon}(\theta,G\boldsymbol{g}) = \hat{\varepsilon}(\theta,\boldsymbol{g}), \qquad \forall\, G \in \mathcal{G}_\kappa \subseteq \mathcal{O}(V).$$

Note that the function $\hat{\boldsymbol{q}}$ and $\hat{\varepsilon}$ are independent of the deformation gradient and are invariant functions with respect to the symmetry group of the body.

It is inconvenient to use these constitutive equations in the spatial description of the energy balance,

$$\rho\,\dot{\varepsilon} + \operatorname{div}\boldsymbol{q} = \rho r, \tag{5.54}$$

because the constitutive equations (5.52) and (5.53) contain the rotational part of the rigid body motion.

We can rewrite the energy equation in the referential description,

$$\rho_\kappa\,\dot{\varepsilon} + \operatorname{Div}\boldsymbol{q}_\kappa = \rho_\kappa r. \tag{5.55}$$

where

$$\boldsymbol{q}_\kappa = |\det F|F^T\boldsymbol{q}, \qquad \rho_\kappa = |\det F|\rho.$$

Since F is orthogonal, $|\det F| = 1$, it follows that

$$\boldsymbol{q}_\kappa = \hat{\boldsymbol{q}}(\theta,\boldsymbol{g}_\kappa), \qquad \varepsilon = \hat{\varepsilon}(\theta,\boldsymbol{g}_\kappa), \tag{5.56}$$

where $\boldsymbol{g}_\kappa = F^T\boldsymbol{g}$ is the referential gradient of temperature. and the constitutive functions $\hat{\boldsymbol{q}}$ and $\hat{\varepsilon}$ are invariant vector and scalar functions with respect to the symmetry group respectively, i.e.,

$$\hat{\boldsymbol{q}}(\theta, Q\boldsymbol{g}_\kappa) = Q\hat{\boldsymbol{q}}(\theta, \boldsymbol{g}_\kappa), \qquad \hat{\varepsilon}(\theta, Q\boldsymbol{g}_\kappa) = \hat{\varepsilon}(\theta, \boldsymbol{g}_\kappa),$$

for all $Q \in \mathcal{G}_\kappa \subseteq \mathcal{O}(V)$.

Remark: For rigid heat conductors, it is usually assumed that the constitutive functions do not depend on the deformation gradient by regarding $F = I$, so that

$$\boldsymbol{q} = \boldsymbol{q}(\theta, \boldsymbol{g}), \qquad \varepsilon = \varepsilon(\theta, \boldsymbol{g}). \tag{5.57}$$

Since $F = I$, the energy equation (5.54) and (5.55) as well as the constitutive equations (5.56) and (5.57) are identical.

However, one can not impose the condition of material objectivity on the constitutive function (5.57) directly, namely,

$$\boldsymbol{q}(\theta, Q\boldsymbol{g}) = Q\boldsymbol{q}(\theta, \boldsymbol{g}), \qquad \forall\, Q \in O(V),$$

since otherwise, it would imply that $\boldsymbol{q}(\theta, \boldsymbol{g})$ is an isotropic function, and consequently, would lead to an absurd conclusion that rigid heat conductors are always isotropic.

CHAPTER 6

Thermodynamic Considerations

6.1 Second law of thermodynamics

In this chapter we shall give a brief consideration of thermodynamic restrictions imposed on constitutive equations. We have already mentioned the first law of thermodynamics, i.e., the energy balance, in Chapter 2. Now we are going to consider the second law for which the essential quantity is the *entropy*,

$$\int_{\mathcal{P}_t} \rho\,\eta\,dv, \tag{6.1}$$

where $\eta(\boldsymbol{x},t)$ is called the specific *entropy density*. Unlike the total energy, the rate of change of total entropy of a body can not be given completely in the form of a balance equation (2.1). There are internal entropy productions in "non-equilibrium" processes.

Entropy production. *For any part $\mathcal{P}\subset\mathcal{B}$, the entropy production $\sigma(\mathcal{P},t)$ is given by*

$$\sigma(\mathcal{P},t) = \frac{d}{dt}\int_{\mathcal{P}_t} \rho\,\eta\,dv + \int_{\partial\mathcal{P}_t} \boldsymbol{\Phi}\cdot\boldsymbol{n}\,da - \int_{\mathcal{P}_t} \rho\,s\,dv.$$

We call $\boldsymbol{\Phi}(\boldsymbol{x},t)$ the *entropy flux* and s the external *entropy supply density.* Although entropy is not a quantity associated with some easily measurable physical quantities, its existence is usually inferred from some more fundamental hypotheses concerning thermal behaviors of material bodies, usually known as the second law of thermodynamics. We choose to accept the existence of entropy and state the consequence of such hypotheses directly by saying that the entropy production is a non-negative quantity.

Second law of thermodynamics. *The following entropy inequality must hold for any part $\mathcal{P}\subset\mathcal{B}$:*

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\,\eta\,dv + \int_{\partial\mathcal{P}_t} \boldsymbol{\Phi}\cdot\boldsymbol{n}\,da - \int_{\mathcal{P}_t} \rho\,s\,dv \ge 0. \tag{6.2}$$

Comparing with the general balance equation (2.1) by setting

$$\psi = \rho\,\eta, \qquad \Phi_\psi = \boldsymbol{\Phi}, \qquad \sigma_\psi = \rho\,s,$$

we have the following local form of the entropy inequality,

$$\rho\dot{\eta} + \operatorname{div}\boldsymbol{\Phi} - \rho s \ge 0. \tag{6.3}$$

6.2 Entropy principle

One of the principal objectives of continuum mechanics is to determine or predict the behavior of a body once the external causes are specified. Mathematically, this amounts to solve initial boundary value problems governed by the balance laws of mass, linear momentum and energy,

$$\begin{aligned} &\dot{\rho} + \rho \operatorname{div} \dot{\boldsymbol{x}} = 0, \\ &\rho \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho \boldsymbol{b}, \\ &\rho \dot{\varepsilon} + \operatorname{div} \boldsymbol{q} - T \cdot \operatorname{grad} \dot{\boldsymbol{x}} = \rho r, \end{aligned} \tag{6.4}$$

when the external supplies $\boldsymbol{b}$ and r are given.

The governing field equations are obtained, for the determination of the fields of the density $\rho(\boldsymbol{X}, t)$, the motion $\chi(\boldsymbol{X}, t)$, and the temperature $\theta(\boldsymbol{X}, t)$, after introducing the constitutive relations for T, ε, and $\boldsymbol{q}$, into the balance laws (6.4). Any solution $\{\rho(\boldsymbol{X}, t), \chi(\boldsymbol{X}, t), \theta(\boldsymbol{X}, t)\}$ of the field equations is called a *thermodynamic process*.

On the other hand, the behavior of a body must also obey the second law of thermodynamics, *i.e.*, a thermodynamic process must also satisfy the entropy inequality (6.3). Following the idea set forth in the fundamental memoir of Coleman and Noll [5], the second law of thermodynamics plays an essential role in constitutive theories of continuum mechanics.

Entropy principle. *It is required that constitutive relations be such that the entropy inequality is satisfied identically for any thermodynamic process.*

From this point of view, like the principle of material objectivity and material symmetry, the entropy principle also imposes restrictions on constitutive functions. To find such restrictions is one of the major task in modern continuum thermodynamics. We shall illustrate the procedures of exploiting the entropy principle for thermoelastic materials.

Remark: Motivated by the results of classical thermostatics, it is often assumed that the entropy flux and the entropy supply are proportional to the heat flux and the heat supply respectively. Moreover, both proportional constants are assumed to be the reciprocal of the *absolute temperature* θ,

$$\boldsymbol{\Phi} = \frac{1}{\theta} \boldsymbol{q}, \qquad s = \frac{1}{\theta} r. \tag{6.5}$$

The resulting entropy inequality is called the *Clausius-Duhem inequality*,

$$\rho \dot{\eta} + \operatorname{div} \frac{\boldsymbol{q}}{\theta} - \rho \frac{r}{\theta} \geq 0. \tag{6.6}$$

Exploitation of entropy principle based on the Clausius-Duhem inequality has been widely adopted in the development of modern continuum thermodynamics following the simple Coleman-Noll procedure. The main assumptions (6.5) while seem to be plausible in all classical theories of continuum mechanics, are not particularly well motivated for materials in general. In fact, the relation (6.5) is known to be inconsistent with the results from the kinetic theory of ideal gases and is also found to be inappropriate to account for thermodynamics of diffusion. Therefore, we shall not abide this assumptions in the present considerations. Exploitation of the entropy principle based on the entropy inequality in its general form (6.3) has been proposed by Müller [15, 16] and the method of Lagrange multipliers proposed by Liu [10] greatly facilitates its procedure.

6.3 Thermodynamics of elastic materials

We shall now exploit the entropy principle for thermoelastic materials, following the Müller-Liu procedure. Although the results will not be different from those following the much simpler Coleman-Noll procedure in this case, it is a typical example to illustrate the method of Lagrange multipliers for the exploitation the thermodynamic restrictions in general.

First of all, Müller proposed that if the body is free of external supplies, the entropy supply must also vanish, which is certainly much weaker than the assumptions (6.5). Since constitutive relations do not depend on the external supplies, in exploiting thermodynamic restrictions it suffices to consider only supply-free bodies.

It is more convenience to use referential description for elastic bodies. For supply-free bodies, from (2.36) we have the following balance laws,

$$\begin{aligned} &\rho = |J|^{-1}\rho_\kappa, \\ &\rho_\kappa \ddot{\boldsymbol{x}} - \operatorname{Div} T_\kappa = 0, \\ &\rho_\kappa \dot{\varepsilon} + \operatorname{Div} \boldsymbol{q}_\kappa - T_\kappa \cdot \dot{F} = 0, \end{aligned} \tag{6.7}$$

and the entropy inequality,

$$\rho_\kappa \dot{\eta} + \operatorname{Div} \boldsymbol{\Phi}_\kappa \geq 0, \tag{6.8}$$

where $T_\kappa = JTF^{-T}$ and $\boldsymbol{q}_\kappa = JF^{-1}\boldsymbol{q}$ (see (2.37)) are the Piola–Kirchhoff stress tensor and the material heat flux respectively. In analogy to the material heat flux, the material entropy flux $\boldsymbol{\Phi}_\kappa$ is similarly defined.

The constitutive relations for thermoelastic materials (5.24) can be written as

$$\begin{aligned} T_\kappa &= \widehat{T}(F,\theta,\boldsymbol{g}_\kappa), & \eta \;&= \hat{\eta}(F,\theta,\boldsymbol{g}_\kappa), \\ \boldsymbol{q}_\kappa &= \widehat{\boldsymbol{q}}(F,\theta,\boldsymbol{g}_\kappa), & \boldsymbol{\Phi}_\kappa &= \widehat{\boldsymbol{\Phi}}(F,\theta,\boldsymbol{g}_\kappa). \\ \varepsilon \;&= \hat{\varepsilon}(F,\theta,\boldsymbol{g}_\kappa), \end{aligned} \tag{6.9}$$

where $\boldsymbol{g}_\kappa = \nabla\theta$ and $F = \nabla\chi$ are the gradients in material coordinate. Meanwhile, θ will be regarded as an *empirical* temperature.

Note that the density field $\rho(\boldsymbol{X},t)$ is completely determined by the motion $\chi(\boldsymbol{X},t)$ and the density $\rho_\kappa(\boldsymbol{X})$ in the reference configuration. Therefore, we can define a *thermodynamic process* as a solution $\{\chi(\boldsymbol{X},t),\theta(\boldsymbol{X},t)\}$ of the field equations by introducing the constitutive relations for T_κ, $\boldsymbol{q}_\kappa$ and ε into the balance laws $(6.7)_{2,3}$ of linear momentum and energy.

The entropy principle requires that the entropy inequality (6.8) must hold for any thermodynamic process $\{\chi(\boldsymbol{X},t),\theta(\boldsymbol{X},t)\}$. This requirement can be stated in a different way, namely, the fields that satisfy the entropy inequality are constrained by the requirement that they must be solutions of the field equations. Following Liu [10] we can take care of this requirement by the use of Lagrange multipliers much like that in the classical problems of finding the extremum with constraints[1]:

Method of Lagrange multipliers. *There exist Lagrange multipliers Λ^v and Λ^ε such that the inequality*

$$\rho_\kappa \dot\eta + \operatorname{Div} \boldsymbol{\Phi}_\kappa - \Lambda^v(\rho_\kappa \ddot{\boldsymbol{x}} - \operatorname{Div} T_\kappa) - \Lambda^\varepsilon(\rho_\kappa \dot\varepsilon + \operatorname{Div} \boldsymbol{q}_\kappa - T_\kappa \cdot \dot F) \geq 0 \tag{6.10}$$

is valid under no constraints, i.e., it must hold for any fields $\{\chi(\boldsymbol{X}, t), \theta(\boldsymbol{X}, t)\}$. Moreover, the Lagrange multipliers are functions of $(F, \theta, \nabla\theta)$.

Note that after introducing the constitutive relations (6.9) into (6.10), the inequality assumed the following form:

$$\sum_a S_a \cdot X_a + \sigma \geq 0, \qquad a = 1, \cdots, 40. \tag{6.11}$$

where $X_a = (\dot\theta, \ddot\chi, \nabla\dot\theta, \nabla\dot\chi, \nabla(\nabla\theta), \nabla(\nabla\chi))$, while S_a and σ are functions of $(\theta, \nabla\theta, \nabla\chi)$. Here, we have use the notation, $\nabla F = \nabla(\nabla\chi)$, to emphasize the symmetry of the second gradient.

Since the inequality (6.11) must hold for any functions $\chi(\boldsymbol{X}, t)$ and $\theta(\boldsymbol{X}, t)$, the values of $(\theta, \nabla\theta, \nabla\chi)$, as well as the derivatives $(\dot\theta, \ddot\chi, \nabla\dot\theta, \nabla\dot\chi, \nabla(\nabla\theta), \nabla(\nabla\chi))$ in (6.11) can be given arbitrarily at any point and any instant.

Note that the inequality (6.11) is linear in X_a, and the values of X_a can be given independently of the values of S_a and σ. This implies that S_a (respecting the part involved with the symmetry of the second gradients in the corresponding X_a) must vanish, otherwise, it is possible to choose some values of X_a such that the inequality is violated.

First of all, from (6.10), we note that the coefficient of $\ddot\chi$ is $\rho_\kappa \Lambda^v$, therefore, we conclude that

$$\Lambda^v = 0,$$

and the inequality (6.10) becomes

$$\rho_\kappa(\dot\eta - \Lambda^\varepsilon \dot\varepsilon) + (\operatorname{Div} \boldsymbol{\Phi}_\kappa - \Lambda^\varepsilon \operatorname{Div} \boldsymbol{q}_\kappa) + \Lambda^\varepsilon T_\kappa \cdot \dot F \geq 0. \tag{6.12}$$

Since both η and ε as well as Λ^ε are functions of $(F, \theta, \nabla\theta)$, we can write

$$\rho_\kappa(\dot\eta - \Lambda^\varepsilon \dot\varepsilon) = H_\theta \dot\theta + H_g \cdot \nabla\dot\theta + H_F \cdot \dot F,$$

where H_θ, H_g and H_F are functions of $(F, \theta, \nabla\theta)$. The linearity of the inequality (6.12) in $(\dot\theta, \nabla\dot\theta, \dot F)$ then leads to

$$H_\theta = 0, \qquad H_g = 0, \qquad H_F = -\Lambda^\varepsilon T_\kappa. \tag{6.13}$$

The inequality (6.12) now reduces to $\operatorname{Div} \boldsymbol{\Phi}_\kappa - \Lambda^\varepsilon \operatorname{Div} \boldsymbol{q}_\kappa \geq 0$, which after introducing the constitutive function (6.9) can be written in the form,

$$\Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial\theta} - \Lambda^\varepsilon \frac{\partial \boldsymbol{q}_\kappa}{\partial\theta}\Big) \cdot \nabla\theta + G \cdot \nabla(\nabla\theta) + M \cdot \nabla(\nabla\chi) \geq 0, \tag{6.14}$$

[1] For general information on the applicability of the method of Lagrange multipliers, please refer to [10] or Chap. 7 of [14].

where in component forms,

$$G_{\alpha\beta} = \frac{\partial \hat{\Phi}_\alpha}{\partial \theta_{,\beta}} - \Lambda^\varepsilon \frac{\partial \hat{q}_\alpha}{\partial \theta_{,\beta}} \qquad M^i_{\alpha\beta} = \frac{\partial \hat{\Phi}_\alpha}{\partial \chi^i_{\ ,\beta}} - \Lambda^\varepsilon \frac{\partial \hat{q}_\alpha}{\partial \chi^i_{\ ,\beta}}.$$

By the linearity of (6.14) in $\nabla(\nabla\theta)$ and $\nabla(\nabla\chi)$, and the symmetry of the second gradients, the symmetric parts of the coefficients must vanish,

$$G_{\alpha\beta} + G_{\beta\alpha} = 0, \qquad M^i_{\alpha\beta} + M^i_{\beta\alpha} = 0. \tag{6.15}$$

In the remaining of this section, we shall prove further that $M^i_{\alpha\beta} = 0$. In order to do this, we need to invoke the *condition of material objectivity* (see (3.35)) of the heat flux, which for elastic materials, can be expressed as

$$\boldsymbol{q}(QF, \theta, Q\boldsymbol{g}) = Q\,\boldsymbol{q}(F, \theta, \boldsymbol{g}) \qquad \forall\, Q \in \mathcal{O}(V),$$

where $\boldsymbol{g} = \operatorname{grad}\theta = F^{-T}\boldsymbol{g}_\kappa$ by (1.23). Let

$$\boldsymbol{q}(F, \theta, \boldsymbol{g}) = \bar{\boldsymbol{q}}(F, \theta, \boldsymbol{g}_\kappa), \tag{6.16}$$

then it is an easy exercise to show that

$$\bar{\boldsymbol{q}}(QF, \theta, \boldsymbol{g}_\kappa) = Q\,\bar{\boldsymbol{q}}(F, \theta, \boldsymbol{g}_\kappa) \qquad \forall\, Q \in \mathcal{O}(V).$$

Taking $Q = R^T$, with R being the rotational part of the polar decomposition $F = RU$, we obtain

$$\bar{\boldsymbol{q}}(F, \theta, \boldsymbol{g}_\kappa) = R\,\bar{\boldsymbol{q}}(U, \theta, \boldsymbol{g}_\kappa). \tag{6.17}$$

By the definition of material heat flux and using the right Cauchy-Green tensor $C = F^TF$, the constitutive relation for the material heat flux (and similarly for material entropy flux) can be written as

$$\boldsymbol{q}_\kappa = \widetilde{\boldsymbol{q}}_\kappa(C, \theta, \boldsymbol{g}_\kappa), \qquad \boldsymbol{\Phi}_\kappa = \widetilde{\boldsymbol{\Phi}}_\kappa(C, \theta, \boldsymbol{g}_\kappa), \tag{6.18}$$

where $C = F^TF$ is the right Cauchy-Green tensor (see e.g. [14, 16, 22]).

On the other hand, since $C = F^TF$, we have, for any tensor A,

$$\frac{\partial \widehat{\boldsymbol{q}}_\kappa}{\partial F}[A] = \frac{\partial \widetilde{\boldsymbol{q}}_\kappa}{\partial C}[A^TF + F^TA] = 2F\frac{\partial \widetilde{\boldsymbol{q}}_\kappa}{\partial C}[A],$$

because C is symmetric and accordingly the gradient must be symmetrized. We shall write the gradient after being symmetrized in components simply as

$$\frac{\partial \widetilde{q}_\alpha}{\partial C_{\beta\gamma}} \qquad \text{for} \qquad \frac{1}{2}\Big(\frac{\partial \widetilde{q}_\alpha}{\partial C_{\beta\gamma}} + \frac{\partial \widetilde{q}_\alpha}{\partial C_{\gamma\beta}}\Big),$$

and hence

$$\frac{\partial \widehat{q}_\alpha}{\partial F_{i\beta}} = 2\,F^i_{\ \gamma}\frac{\partial \widetilde{q}_\alpha}{\partial C_{\beta\gamma}}.$$

Likewise, similar relations are valid for the material entropy flux $\widehat{\boldsymbol{\Phi}}_\kappa$.

Let

$$\tilde{M}^{\gamma}_{\alpha\beta} = \frac{\partial \tilde{\Phi}_{\alpha}}{\partial C_{\beta\gamma}} - \Lambda^{\varepsilon} \frac{\partial \tilde{q}_{\alpha}}{\partial C_{\beta\gamma}},$$

then the relation (6.15)$_2$, $M^i_{\alpha\beta} + M^i_{\beta\alpha} = 0$, becomes

$$F^i{}_{\gamma}(\tilde{M}^{\gamma}_{\alpha\beta} + \tilde{M}^{\gamma}_{\beta\alpha}) = 0.$$

Since F is non-singular, it implies that $\tilde{M}^{\gamma}_{\alpha\beta} = -\tilde{M}^{\gamma}_{\beta\alpha}$ and for the symmetrization of the gradient in $C_{\beta\gamma}$, it follows that $\tilde{M}^{\gamma}_{\alpha\beta} = \tilde{M}^{\beta}_{\alpha\gamma}$. Therefore, we have

$$\tilde{M}^{\gamma}_{\alpha\beta} = \tilde{M}^{\beta}_{\alpha\gamma} = -\tilde{M}^{\beta}_{\gamma\alpha} = -\tilde{M}^{\alpha}_{\gamma\beta} = \tilde{M}^{\alpha}_{\beta\gamma} = \tilde{M}^{\gamma}_{\beta\alpha} = -\tilde{M}^{\gamma}_{\alpha\beta},$$

and hence $\tilde{M}^{\gamma}_{\alpha\beta} = 0$, i.e.,

$$\frac{\partial \tilde{\boldsymbol{\Phi}}_{\kappa}}{\partial C} - \Lambda^{\varepsilon} \frac{\partial \tilde{\boldsymbol{q}}_{\kappa}}{\partial C} = 0 \quad \text{or} \quad \frac{\partial \widehat{\boldsymbol{\Phi}}_{\kappa}}{\partial F} - \Lambda^{\varepsilon} \frac{\partial \widehat{\boldsymbol{q}}_{\kappa}}{\partial F} = 0. \tag{6.19}$$

Summary of thermodynamic restrictions

The restrictions imposed by the entropy principle on elastic materials in the above exploitation can be summarized below.

The inequality of (6.10) has been reduced successively to (6.12) and (6.14), and finally to the remaining one, which gives the entropy production density σ as

$$\sigma = \Big(\frac{\partial \widehat{\boldsymbol{\Phi}}_{\kappa}}{\partial \theta} - \Lambda^{\varepsilon} \frac{\partial \widehat{\boldsymbol{q}}_{\kappa}}{\partial \theta} \Big) \cdot \boldsymbol{g}_{\kappa} \geq 0. \tag{6.20}$$

The material entropy flux and heat flux must satisfy the relations (6.15)$_1$ and (6.19)$_2$, which can now be written as

$$\Big(\frac{\partial \widehat{\boldsymbol{\Phi}}_{\kappa}}{\partial \widehat{\boldsymbol{g}}_{\kappa}} \Big)_{\text{sym}} = \Lambda^{\varepsilon} \Big(\frac{\partial \widehat{\boldsymbol{q}}_{\kappa}}{\partial \boldsymbol{g}_{\kappa}} \Big)_{\text{sym}}, \qquad \frac{\partial \widehat{\boldsymbol{\Phi}}_{\kappa}}{\partial F} = \Lambda^{\varepsilon} \frac{\partial \widehat{\boldsymbol{q}}_{\kappa}}{\partial F}, \tag{6.21}$$

where $(A)_{\text{sym}}$ denotes the symmetric part of the tensor A. Finally, the relations (6.13) can be summarized in the following differential expression, similar to the Gibbs relation in classical thermodynamics,

$$d\eta = \Lambda^{\varepsilon} \Big(d\varepsilon - \frac{1}{\rho_{\kappa}} T_{\kappa} \cdot dF \Big). \tag{6.22}$$

The relations (6.20), (6.21) and (6.22) are the thermodynamic restrictions for elastic bodies in general. Note that these relations contain one remaining Lagrange multiplier Λ^{ε}, which depends on $(F, \theta, \boldsymbol{g}_{\kappa})$ and so are the other constitutive functions. Further reductions will be considered in the next sections.

Remark: The relations (6.21) and (6.22), as compared to the well-known Gibbs relation, seem to suggest that the assumption (6.5)$_1$, i.e., $\boldsymbol{\Phi}_{\kappa} = (1/\theta)\boldsymbol{q}_{\kappa}$, might be valid in this case. Unfortunately, a rigorous proof has not been available for elastic bodies in general.

6.4 Elastic materials of Fourier type

An elastic material will be called a material of Fourier type, if the heat flux and the entropy flux depend linearly on the temperature gradient,

$$\boldsymbol{q}_\kappa = -K(F,\theta)\,\boldsymbol{g}_\kappa, \qquad \boldsymbol{\Phi}_\kappa = -P(F,\theta)\,\boldsymbol{g}_\kappa. \tag{6.23}$$

We shall further assume that the *thermal conductivity* tensor K and the tensor P be symmetric.

It follows from the relation $(6.21)_1$ that $P = \Lambda^\varepsilon K$, which gives immediately that

$$\boldsymbol{\Phi}_\kappa = \Lambda^\varepsilon \boldsymbol{q}_\kappa. \tag{6.24}$$

Taking the gradient of (6.24) with respect to $\boldsymbol{g}_\kappa$ and using again the relation $(6.21)_1$, we obtain

$$\frac{\partial \Lambda^\varepsilon}{\partial \boldsymbol{g}_\kappa} \cdot \boldsymbol{q}_\kappa = \operatorname{tr}\Big(\frac{\partial \boldsymbol{\Phi}_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \boldsymbol{q}_\kappa}{\partial \boldsymbol{g}_\kappa}\Big) = 0,$$

from which it implies that Λ^ε must be independent of $\boldsymbol{g}_\kappa$, since $\boldsymbol{q}_\kappa$ does not vanish in general. Similarly, by taking the gradient with respect to F and using the relation $(6.21)_2$, it follows immediately that Λ^ε must be independent of F. Therefore, we have

$$\Lambda^\varepsilon = \Lambda(\theta),$$

and by $(6.23)_1$, the entropy production density (6.20) becomes

$$\sigma = -\Big(\frac{\partial \Lambda^\varepsilon}{\partial \theta}\Big)(\boldsymbol{g}_\kappa \cdot K\,\boldsymbol{g}_\kappa) \geq 0.$$

Since the entropy production does not vanish identically in heat conducting bodies, we require that $\partial \Lambda^\varepsilon/\partial\theta \neq 0$. Consequently, $\Lambda(\theta)$ depends monotonically on θ and hence $\Lambda(\theta)$ can also be taken as a temperature measure referred to as the *coldness*. By comparison of (6.22) with the classical Gibbs relation, we can now conveniently choose θ as the *absolute temperature*, and set

$$\Lambda^\varepsilon(\theta) = \frac{1}{\theta}. \tag{6.25}$$

Therefore, we have the following Gibbs relation,

$$d\eta = \frac{1}{\theta}\Big(d\varepsilon - \frac{1}{\rho_\kappa} T_\kappa \cdot dF\Big), \tag{6.26}$$

from which we obtain

$$\frac{\partial \eta}{\partial \theta} = \frac{1}{\theta}\frac{\partial \varepsilon}{\partial \theta}, \qquad \frac{\partial \eta}{\partial \boldsymbol{g}_\kappa} = \frac{1}{\theta}\frac{\partial \varepsilon}{\partial \boldsymbol{g}_\kappa}.$$

The integrability condition, i.e., by taking the mixed partial derivative with respect to θ and $\boldsymbol{g}_\kappa$ from the above two relations, implies that ε and hence η are independent of $\boldsymbol{g}_\kappa$.

By the use of the *free energy* function $\psi(F,\theta)$ defined by

$$\psi = \varepsilon - \theta\eta,$$

the Gibbs relation (6.26) can be rewritten as

$$d\psi = \frac{1}{\rho_\kappa} T_\kappa \cdot dF - \eta \, d\theta,$$

which implies that the following constitutive relations are determined completely by a single scalar function $\psi(F, \theta)$:

$$T_\kappa = \rho_\kappa \frac{\partial \psi}{\partial F}, \qquad \eta = -\frac{\partial \psi}{\partial \theta}, \qquad \varepsilon = \psi - \theta \frac{\partial \psi}{\partial \theta}.$$

Finally, the remaining entropy production inequality takes the form,

$$\sigma = -\frac{1}{\theta^2} (\boldsymbol{q}_\kappa \cdot \boldsymbol{g}_\kappa) = \frac{1}{\theta^2} (\boldsymbol{g}_\kappa \cdot K \, \boldsymbol{g}_\kappa) \geq 0,$$

which implies that the thermal conductivity $K(F, \theta)$ is a positive semi-definite tensor (see (5.34)).

Changing the Piola-Kirchhoff stress tensor T_κ into the the Cauchy stress tensor T, we can rewrite the Gibbs relation (6.26) as

$$d\eta = \frac{1}{\theta} \Big(d\varepsilon - \frac{1}{\rho} T F^{-T} \cdot dF \Big), \tag{6.27}$$

and summarize the above general thermodynamic restrictions in the following proposition:

Proposition. *The stress tensor, the internal energy and the entropy are related to the free energy function* $\psi = \psi(F, \theta)$ *by*

$$T = \rho \frac{\partial \psi}{\partial F} F^T, \qquad \varepsilon = \psi - \theta \frac{\partial \psi}{\partial \theta}, \qquad \eta = -\frac{\partial \psi}{\partial \theta}, \tag{6.28}$$

and the heat flux satisfies the Fourier inequality,

$$\boldsymbol{q}(F, \theta, \boldsymbol{g}) \cdot \boldsymbol{g} \leq 0. \tag{6.29}$$

Moreover, the following entropy flux relation is valid,

$$\boldsymbol{\Phi} = \frac{1}{\theta} \boldsymbol{q}.$$

An elastic material is called *hyperelastic* if there exists a function $\phi(F)$, called a *stored energy* function, such that the stress tensor is given by

$$T = \rho \frac{\partial \phi}{\partial F} F^T \quad \text{or} \quad T_\kappa = \rho_\kappa \frac{\partial \phi}{\partial F}. \tag{6.30}$$

From the relation $(6.28)_1$, it follows that in an isothermal process, an elastic material of Fourier type is a hyperelastic material, with free energy function ψ served as the stored energy function. In particular, for the linear theory considered in Section 5.2, the relation $(6.30)_2$ reduces to (5.10).

6.5 Isotropic elastic materials

In the previous section, we have seen that further evaluations from the general restrictions (6.20), (6.21) and (6.22) rely essentially on the flux relation (6.24), namely, $\boldsymbol{\Phi}_\kappa = \Lambda^\varepsilon \boldsymbol{q}_\kappa$, which can easily be verified for elastic materials of Fourier type. Now, we shall prove that the flux relation is also valid for isotropic elastic materials.

For isotropic materials, the material symmetric group is the full orthogonal group, in particular, from (3.39) for the heat flux $\boldsymbol{q} = \boldsymbol{q}(F, \theta, \boldsymbol{g})$, we have

$$\boldsymbol{q}(FG, \theta, \boldsymbol{g}) = \boldsymbol{q}(F, \theta, \boldsymbol{g}) \qquad \forall\, G \in \mathcal{O}(V),$$

or equivalently from (6.16) with $\boldsymbol{g}_\kappa = F^T \boldsymbol{g}$ and $Q = G^T$,

$$\bar{\boldsymbol{q}}(FQ^T, \theta, Q\boldsymbol{g}_\kappa) = \bar{\boldsymbol{q}}(F, \theta, \boldsymbol{g}_\kappa) \qquad \forall\, Q \in \mathcal{O}(V),$$

which combined with the material objectivity condition (6.17) by use of the polar decomposition, $FQ^T = (RQ^T)(QUQ^T)$, leads to the following condition for material symmetry of isotropic elastic materials,

$$\bar{\boldsymbol{q}}(QUQ^T, \theta, Q\boldsymbol{g}_\kappa) = Q\, \bar{\boldsymbol{q}}(U, \theta, \boldsymbol{g}_\kappa) \qquad \forall\, Q \in \mathcal{O}(V).$$

In terms of material flux (6.18), the material symmetry condition for the material fluxes $\boldsymbol{q}_\kappa$ and $\boldsymbol{\Phi}_\kappa$ can be expressed as

$$\begin{aligned} \tilde{\boldsymbol{q}}_\kappa(QCQ^T, \theta, Q\boldsymbol{g}_\kappa) &= Q\, \tilde{\boldsymbol{q}}_\kappa(C, \theta, \boldsymbol{g}_\kappa), \\ \tilde{\boldsymbol{\Phi}}_\kappa(QCQ^T, \theta, Q\boldsymbol{g}_\kappa) &= Q\, \tilde{\boldsymbol{\Phi}}_\kappa(C, \theta, \boldsymbol{g}_\kappa), \end{aligned} \qquad \forall\, Q \in \mathcal{O}. \tag{6.31}$$

In other words, they are isotropic vector-valued functions of $(C, \theta, \boldsymbol{g}_\kappa)$.

An isotropic function is restricted in its dependence on the independent variables as we have seen in Sect. 4.2. Here, instead of using the explicit representation formula given in (4.14), we shall consider restrictions in the form of differential equations. We need the following lemma:

Lemma. *Let $\mathcal{F}(A)$ be a scalar-valued function of a tensor variable and suppose that $\mathcal{F}(Q) = 0$ for any orthogonal $Q \in \mathcal{O}$. Then the gradient of $\mathcal{F}(A)$ at the identity tensor is symmetric, i.e., for any skew symmetric tensor W,*

$$\nabla_A \mathcal{F}(1)[W] = 0. \tag{6.32}$$

Proof. For any skew symmetric tensor W, we can define[2] a time-dependent orthogonal tensor $Q(t) = \exp tW$. Therefore, $\mathcal{F}(Q(t)) = 0$. By taking the time derivative, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(Q(t)) &= \nabla_A \mathcal{F}(\exp tW)\Big[\frac{d}{dt}\exp tW\Big] \\ &= \nabla_A \mathcal{F}(\exp tW)[W \exp tW] = 0, \end{aligned}$$

which, at $t = 0$, proves the lemma. □

[2] See footnote on page 49.

For the material heat flux $\boldsymbol{q}_\kappa = \widetilde{\boldsymbol{q}}_\kappa(C, \theta, \boldsymbol{g}_\kappa)$, let us consider, for any constant vector $\boldsymbol{a}$,

$$\mathcal{F}(A) = \boldsymbol{a} \cdot (\widetilde{\boldsymbol{q}}_\kappa(ACA^T, \theta, A\boldsymbol{g}_\kappa) - A\,\widetilde{\boldsymbol{q}}_\kappa(C, \theta, \boldsymbol{g}_\kappa)).$$

Then from the condition (6.31), it follows that $\mathcal{F}(Q) = 0$ for any $Q \in \mathcal{O}$ and hence the above Lemma implies that

$$\nabla_A \mathcal{F}(1)[W] = \boldsymbol{a} \cdot \frac{\partial \widetilde{\boldsymbol{q}}_\kappa}{\partial C}[WC - CW] + \boldsymbol{a} \cdot \frac{\partial \widetilde{\boldsymbol{q}}_\kappa}{\partial \boldsymbol{g}_\kappa} W \boldsymbol{g}_\kappa - \boldsymbol{a} \cdot W \widetilde{\boldsymbol{q}}_\kappa = 0.$$

In analogy, we have a similar equation for the material entropy flux $\boldsymbol{\Phi}_\kappa = \widetilde{\boldsymbol{\Phi}}_\kappa(C, \theta, \boldsymbol{g}_\kappa)$.

Now, let $\boldsymbol{k} = \boldsymbol{\Phi}_\kappa - \Lambda^\varepsilon \boldsymbol{q}_\kappa$. From the above relations for both $\widetilde{\boldsymbol{\Phi}}_\kappa$ and $\widetilde{\boldsymbol{q}}_\kappa$, and the relation $(6.19)_1$, we have

$$\boldsymbol{a} \cdot W \boldsymbol{k} = \boldsymbol{a} \cdot \widetilde{G} W \boldsymbol{g}_\kappa, \tag{6.33}$$

where

$$\widetilde{G} = \frac{\partial \widetilde{\boldsymbol{\Phi}}_\kappa}{\partial \boldsymbol{g}_\kappa} - \Lambda^\varepsilon \frac{\partial \widetilde{\boldsymbol{q}}_\kappa}{\partial \boldsymbol{g}_\kappa}.$$

Since it holds for any vector $\boldsymbol{a}$ and any skew symmetric tensor W, in particular, by taking alternatively,

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{e}_1, & W &= \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_2, \\ \boldsymbol{a} &= \boldsymbol{e}_2, & W &= \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_3, \\ \boldsymbol{a} &= \boldsymbol{e}_3, & W &= \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1, \end{aligned}$$

and knowing from $(6.15)_1$ that $\widetilde{G}$ is skew symmetric, we obtain

$$\widetilde{G}_{12} g_3 + \widetilde{G}_{31} g_2 = 0, \qquad \widetilde{G}_{23} g_1 + \widetilde{G}_{12} g_3 = 0, \qquad \widetilde{G}_{31} g_2 + \widetilde{G}_{23} g_1 = 0.$$

Summing of the last three equations gives $\widetilde{G}_{12} g_3 + \widetilde{G}_{23} g_1 + \widetilde{G}_{31} g_2 = 0$, which implies immediately that

$$\widetilde{G}_{12} g_3 = 0, \qquad \widetilde{G}_{23} g_1 = 0, \qquad \widetilde{G}_{31} g_2 = 0.$$

Since $\boldsymbol{g}_\kappa = (g_1, g_2, g_3)$ does not vanish in general, it follows that $\widetilde{G}_{12} = \widetilde{G}_{23} = \widetilde{G}_{31} = 0$ or the skew symmetric tensor $\widetilde{G} = 0$. Therefore, by (6.33), since W is arbitrary, $\boldsymbol{k}$ must vanish and we have proved the flux relation[3],

$$\boldsymbol{\Phi}_\kappa = \Lambda^\varepsilon \boldsymbol{q}_\kappa, \qquad \Lambda^\varepsilon = \Lambda^\varepsilon(\theta).$$

The second relation is a consequence of (6.19) and $G = 0$. □

The remaining evaluations can be done in the same manner given in the previous section. In particular, we can also identify the Lagrange multiplier Λ^ε as the reciprocal of the absolute temperature,

$$\Lambda^\varepsilon = \frac{1}{\theta},$$

and the general results given in the Proposition (on page 80) as well as the Gibbs relation (6.27) are valid for isotropic elastic materials in general, except that the heat flux $\boldsymbol{q}_\kappa$ is not restricted to the case of Fourier type, instead, it is an isotropic vector-valued function of $(C, \theta, \boldsymbol{g}_\kappa)$.

[3] For similar flux relations involving more than one vector and one tensor variables, see [13].

Remark Suppose the entropy flux relation

$$\boldsymbol{\Phi}_\kappa = \Lambda^\varepsilon \boldsymbol{q}_\kappa$$

is valid for each of the two bodies I and II in thermal contact, then at the contact surface, the wall, the following conditions hold,

$$\boldsymbol{q}_\kappa^I \cdot \boldsymbol{n}_\kappa = \boldsymbol{q}_\kappa^{II} \cdot \boldsymbol{n}_\kappa, \qquad \boldsymbol{\Phi}_\kappa^I \cdot \boldsymbol{n}_\kappa = \boldsymbol{\Phi}_\kappa^{II} \cdot \boldsymbol{n}_\kappa,$$

where $\boldsymbol{n}_\kappa$ is the unit normal at the wall. Since from the entropy flux relation,

$$\boldsymbol{\Phi}_\kappa^I = \Lambda^{\varepsilon I} \boldsymbol{q}_\kappa^I, \qquad \boldsymbol{\Phi}_\kappa^{II} = \Lambda^{\varepsilon II} \boldsymbol{q}_\kappa^{II},$$

it follows that the Lagrange multiplier Λ^ε is continuous across the wall,

$$\Lambda^{\varepsilon I} = \Lambda^{\varepsilon II}.$$

This continuity property duely suggests that the Lagrange multiplier Λ^ε be interpreted as a measure of *thermodynamic* temperature. It is referred to as the *coldness* by Müller [16].

Elastic fluids

An elastic fluid (see (4.25)) is an isotropic material, which depends on the deformation gradient F only through its dependence on $\det F$, or equivalently on the density ρ. Consequently, the relation (6.27) reduces to the well-known Gibbs relation for fluids in classical thermostatics:

$$d\eta = \frac{1}{\theta}\Big(d\varepsilon - \frac{p}{\rho^2} d\rho\Big), \tag{6.34}$$

and

$$T = T(\rho, \theta), \qquad \boldsymbol{q} = \boldsymbol{q}(\rho, \theta, \boldsymbol{g}), \qquad \varepsilon = \varepsilon(\rho, \theta).$$

From the representation formula for isotropic functions (as a special case of (4.14)), we have

$$T = -p(\rho, \theta)\, 1, \qquad \boldsymbol{q} = -\kappa(\rho, \theta)\, \boldsymbol{g},$$

where p is the pressure and the (scalar) thermal conductivity κ is non-negative.

6.6 Thermodynamic stability

Another important concept associated with the entropy inequality is the thermodynamic stability of a material body. To illustrate the basic ideas let us consider a supply-free body occupying a region $\mathcal{V}$ with a fixed adiabatic boundary. We have

$$\boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{q} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{\Phi} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\mathcal{V},$$

and hence the entropy inequality (6.2) and the energy balance (2.31) become

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\eta \, dv \geq 0, \qquad \frac{d}{dt}\int_{\mathcal{V}} \rho(\varepsilon + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\, dv = 0. \tag{6.35}$$

In other words, the total entropy must increase in time while the total energy remains constant for a body with fixed adiabatic boundary. Statements of this kind are usually called *stability criteria* as we shall explain in the following example.

Stable equilibrium state. *We say that an equilibrium state is stable if any small disturbance away from it will eventually disappear and thus the original state will be restored.*

Suppose that the region $\mathcal{V}$ is occupied by an elastic fluid in an equilibrium state at rest with constant mass density ρ_o and internal energy density ε_o. Now let us consider a small disturbance from the equilibrium state at the initial time such that

$$\rho(\boldsymbol{x},0) = \hat{\rho}(\boldsymbol{x}), \quad \varepsilon(\boldsymbol{x},0) = \hat{\varepsilon}(\boldsymbol{x}), \quad \boldsymbol{v}(\boldsymbol{x},0) = \mathbf{0},$$

and $|\hat{\rho} - \rho_o|$ and $|\hat{\varepsilon} - \varepsilon_o|$ are small quantities. If we assume that the original state is stable then the perturbed state will eventually return to the original state at later time. Therefore since the total entropy must increase we conclude that

$$\int_{\mathcal{V}} \rho_o \eta_o \, dv \geq \int_{\mathcal{V}} \hat{\rho}\hat{\eta} \, dv, \tag{6.36}$$

where $\eta_o = \eta(\varepsilon_o, \rho_o)$ and $\hat{\eta} = \eta(\hat{\varepsilon}, \hat{\rho})$ are the final equilibrium entropy and the perturbed initial entropy. Expanding $\hat{\eta}$ in Taylor series around the equilibrium state, we obtain from (6.36)

$$\begin{aligned}\int_{\mathcal{V}} \Big\{ \eta_o(\hat{\rho} - \rho_o) + \frac{\partial \eta}{\partial \rho}\Big|_o \hat{\rho}(\hat{\rho} - \rho_o) + \frac{\partial \eta}{\partial \varepsilon}\Big|_o \hat{\rho}(\hat{\varepsilon} - \varepsilon_o) + \frac{1}{2}\frac{\partial^2 \eta}{\partial \varepsilon^2}\Big|_o \hat{\rho}(\hat{\varepsilon} - \varepsilon_o)^2 + \\ \frac{\partial^2 \eta}{\partial \varepsilon \partial \rho}\Big|_o \hat{\rho}(\hat{\varepsilon} - \varepsilon_o)(\hat{\rho} - \rho_o) + \frac{1}{2}\frac{\partial^2 \eta}{\partial \rho^2}\Big|_o \hat{\rho}(\hat{\rho} - \rho_o)^2 \Big\} \, dv + o(3) \leq 0.\end{aligned} \tag{6.37}$$

Since total mass and total energy remain constant, we have

$$\int_{\mathcal{V}} (\hat{\rho} - \rho_o) \, dv = 0, \qquad \int_{\mathcal{V}} (\hat{\rho}\hat{\varepsilon} - \rho_o \varepsilon_o) \, dv = 0,$$

and hence

$$\begin{aligned}\int_{\mathcal{V}} \hat{\rho}(\hat{\varepsilon} - \varepsilon_o) \, dv &= \int_{\mathcal{V}} (\hat{\rho}\hat{\varepsilon} - \rho_o \varepsilon_o) \, dv - \varepsilon_o \int_{\mathcal{V}} (\hat{\rho} - \rho_o) \, dv = 0, \\ \int_{\mathcal{V}} \hat{\rho}(\hat{\rho} - \rho_o) \, dv &= \int_{\mathcal{V}} (\hat{\rho} - \rho_o)^2 dv = \int_{\mathcal{V}} \frac{\hat{\rho}}{\rho_o} (\hat{\rho} - \rho_o)^2 dv + o(3).\end{aligned}$$

Therefore, up to the second order terms (6.37) becomes

$$\int_{\mathcal{V}} \Big\{ \frac{1}{2}\frac{\partial^2 \eta}{\partial \varepsilon^2}\Big|_o \hat{\rho}(\hat{\varepsilon} - \varepsilon_o)^2 + \frac{\partial^2 \eta}{\partial \varepsilon \partial \rho}\Big|_o \hat{\rho}(\hat{\varepsilon} - \varepsilon_o)(\hat{\rho} - \rho_o) + \Big(\frac{1}{2}\frac{\partial^2 \eta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \eta}{\partial \rho} \Big)\Big|_o \hat{\rho}(\hat{\rho} - \rho_o)^2 \Big\} \, dv \leq 0.$$

By the mean value theorem for integrals, it reduces to

$$\Big\{ \frac{\partial^2 \eta}{\partial \varepsilon^2}\Big|_o (\varepsilon^* - \varepsilon_o)^2 + 2\frac{\partial^2 \eta}{\partial \varepsilon \partial \rho}\Big|_o (\varepsilon^* - \varepsilon_o)(\rho^* - \rho_o) + \Big(\frac{\partial^2 \eta}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial \eta}{\partial \rho} \Big)\Big|_o (\rho^* - \rho_o)^2 \Big\} \frac{\rho^* V}{2} \leq 0,$$

where V is the volume of the region $\mathcal{V}$ while $\rho^* = \hat{\rho}(\boldsymbol{x}^*)$ and $\varepsilon^* = \hat{\varepsilon}(\boldsymbol{x}^*)$ for some point $\boldsymbol{x}^*$ in $\mathcal{V}$.

Since the non-positiveness of the above quadratic form must hold for any small disturbance and ρ_o and ε_o are arbitrary, it follows that the matrix

$$\begin{bmatrix} \dfrac{\partial^2\eta}{\partial\varepsilon^2} & \dfrac{\partial^2\eta}{\partial\varepsilon\partial\rho} \\ \dfrac{\partial^2\eta}{\partial\varepsilon\partial\rho} & \dfrac{\partial^2\eta}{\partial\rho^2}+\dfrac{2}{\rho}\dfrac{\partial\eta}{\partial\rho} \end{bmatrix}$$

must be negative semi-definite, or equivalently

$$\frac{\partial^2\eta}{\partial\varepsilon^2}\le 0, \qquad \frac{\partial^2\eta}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial\eta}{\partial\rho}\le 0, \tag{6.38}$$

$$\frac{\partial^2\eta}{\partial\varepsilon^2}\left(\frac{\partial^2\eta}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial\eta}{\partial\rho}\right)-\left(\frac{\partial^2\eta}{\partial\varepsilon\partial\rho}\right)^2\ge 0. \tag{6.39}$$

In these relations partial derivatives are taken with respect to the variables (ε,ρ).

In order to give more suggestive meanings to the above conditions for stability, we shall reiterate them in terms of the independent variables (ρ,θ). In making this change of variables we must admit the invertibility of $\varepsilon(\rho,\theta)$ with respect to the temperature, *i.e.* $\partial\varepsilon/\partial\theta\neq 0$. To avoid confusions variables held constant in partial differentiations will be indicated. Two relations frequently used in thermostatics for change of variables are given below.

Lemma. *Let u, v, and w be three variables and there is a relation between them so that we have the functional relations: $u=u(v,w)$, $v=v(u,w)$, and $w=w(u,v)$. Then*

$$\frac{\partial u}{\partial v}\Big|_w\frac{\partial v}{\partial u}\Big|_w=1, \qquad \frac{\partial u}{\partial v}\Big|_w\frac{\partial v}{\partial w}\Big|_u\frac{\partial w}{\partial u}\Big|_v=-1. \tag{6.40}$$

The proof follows immediately by writing $u=u(v(u,w),w)=f(u,w)$ and computing the partial derivatives of $f(u,w)$ with respect to u and w respectively.

From the Gibbs relation (6.34), we have

$$\frac{\partial\eta}{\partial\varepsilon}\Big|_\rho=\frac{1}{\theta}, \qquad \frac{\partial\eta}{\partial\rho}\Big|_\varepsilon=-\frac{p}{\theta\,\rho^2},$$

which further imply the integrability condition for η,

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\theta}\right)\Big|_\varepsilon=-\frac{1}{\rho^2}\frac{\partial}{\partial\varepsilon}\left(\frac{p}{\theta}\right)\Big|_\rho.$$

By the use of (6.40) and the above relations, we then obtain

$$\begin{aligned}
\frac{\partial^2\eta}{\partial\varepsilon^2}\Big|_\rho &= -\frac{1}{\theta^2}\frac{\partial\varepsilon}{\partial\theta}\Big|_\rho^{-1}, \\
\frac{\partial^2\eta}{\partial\varepsilon\partial\rho} &= \frac{1}{\theta^2}\frac{\partial\varepsilon}{\partial\theta}\Big|_\rho^{-1}\frac{\partial\varepsilon}{\partial\rho}\Big|_\theta, \\
\frac{\partial^2\eta}{\partial\rho^2}\Big|_\varepsilon+\frac{2}{\rho}\frac{\partial\eta}{\partial\rho}\Big|_\varepsilon &= -\frac{1}{\rho^2\theta}\frac{\partial p}{\partial\rho}\Big|_\theta-\frac{1}{\theta^2}\frac{\partial\varepsilon}{\partial\theta}\Big|_\rho^{-1}\frac{\partial\varepsilon}{\partial\rho}\Big|_\theta^2.
\end{aligned}$$

Therefore, in terms of the variables (ρ, θ), the stability condition $(6.38)_1$ reduces to

$$\frac{\partial \varepsilon}{\partial \theta} > 0, \tag{6.41}$$

since we have already admitted that $\partial\varepsilon/\partial\theta \neq 0$. The condition (6.39) after simplifications then reduces to

$$\frac{\partial p}{\partial \rho} \geq 0, \tag{6.42}$$

The remaining condition $(6.38)_2$ is merely a consequence of (6.42). Obviously, these conditions are restrictions on the constitutive functions $\varepsilon(\rho, \theta)$ and $p(\rho, \theta)$.

We have shown that the stability of equilibrium requires that: (1) the *specific heat* at constant volume (equivalently at constant density), $c_v = \partial\varepsilon/\partial\theta$, must be positive (see (5.34)); (2) the isothermal compressibility, $\kappa_T = (\rho\,\partial p/\partial\rho)^{-1}$, must be non-negative.

The condition (6.42) can also be expressed in a different form. Let $\upsilon = 1/\rho$ be the specific volume, the volume per unit mass. Then the Gibbs relation (6.34) can be written as

$$d\psi = -\eta\, d\theta - p\, d\upsilon.$$

and the condition (6.42) implies

$$\left.\frac{\partial^2 \psi}{\partial \upsilon^2}\right|_\theta \geq 0.$$

Therefore, the stability of equilibrium requires that the free energy be a concave-upward function of the specific volume, a well-known result for stability. □

To establish a stability criterion for a material system under a different condition, one may try to find a *decreasing function of time* $\mathcal{A}(t)$ from the balance laws and the entropy inequality in integral forms. Such a function is called the *availability* of the system, since it is the quantity available to the system for its expense in the course toward equilibrium. Such a function is also known as a *Liapounov function* in the stability theory of dynamic systems (see [9]). In the present example, from $(6.35)_1$ one may define the availability $\mathcal{A}$ of the system as

$$\mathcal{A}(t) = -\int_{\mathcal{V}} \rho\eta\, dv, \qquad \frac{d\mathcal{A}}{dt} \leq 0.$$

As a second example, we consider a supply-free body with a fixed isothermal boundary,

$$\boldsymbol{v} = \boldsymbol{0}, \quad \theta = \theta_o \quad \text{on} \ \ \partial\mathcal{V},$$

and assume that the relation $\boldsymbol{\Phi} = \boldsymbol{q}/\theta$ holds. Then the energy balance (2.31) and the entropy inequality (6.2) lead to

$$\frac{d}{dt}\int_{\mathcal{V}} \rho(\varepsilon + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\, dv + \int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n}\, da = 0,$$

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\eta\, dv + \frac{1}{\theta_o}\int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n}\, da \geq 0.$$

Elimination of the terms containing surface integrals from above, gives

$$\frac{d\mathcal{A}}{dt} \leq 0, \quad \mathcal{A}(t) = \int_{\mathcal{V}} \rho(\varepsilon - \theta_o\eta + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\, dv. \tag{6.43}$$

In this manner we have found a decreasing function of time, the availability $\mathcal{A}(t)$, which characterizes the stability for this system. Note that

$$\int_{\mathcal{V}} \rho(\varepsilon - \theta_o\eta)\, dv$$

is the total free energy if $\theta = \theta_o$ throughout the body. Therefore, it follows that for a body with constant uniform temperature in a fixed region the availability $\mathcal{A}$ reduces to the sum of the free energy and the kinetic energy.

Summarizing the above two situations, we can state the following criteria for the stability of equilibrium.

Criteria of thermodynamic stability.

1) *For a body with fixed adiabatic boundary and constant energy, the entropy tends to a maximum in equilibrium.*
2) *For a body with fixed boundary and constant uniform temperature, the sum of the free energy and the kinetic energy tends to a minimum in equilibrium.*

We have seen in this section that thermodynamic stability criteria, like the entropy principle, impose further restrictions on properties of the constitutive functions, namely, specific heat and compressibility must be positive. On the other hand, such criteria, besides being used in analyzing stability of solutions, they are the basic principles for the formulation of equilibrium solutions in terms of minimization (or maximization) problems.

Bibliography

[1] Boehler, J. P.: On irreducible representations for isotropic scalar functions, *ZAMM* **57**, 323-327 (1977).

[2] Coleman, B. D., Markovitz, H., Noll, W.: *Viscometric Flows of Non-Newtonian Fluids*, springer (1966).

[3] Coleman, B. D., Mizel, V. J.: On the general theory of fading memory, *Arch. Rational Mech. Anal.* **29**, 18-31 (1968).

[4] Coleman, B. D., Noll, W.: Foundations of linear viscoelasticity, *Rev. Modern Phys.* **33**, 239-249 (1961).

[5] Coleman, B. D., Noll, W.: The thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rational Mech. Anal.* **13**, 167-178 (1963).

[6] Gurtin, M. E., Martins, L. C.: Cauchy's theorem in classical physics, *Arch. Rational Mech. Anal.* **60**, 305-324 (1976).

[7] Gurtin, M. E., Mizel, V. J., Williams, W. O.: A note on Cauchy's stress theorem, *J. Math. Anal. Appl.* **22**, 398-401 (1968).

[8] Gurtin, M. E., Williams, W. O.: On the inclusion of the complete symmetry group in the unimodular group, *Arch. Rational Mech. Anal.* **23**, 163-172 (1966).

[9] Knops, R. J., Wilkes, E. W.: *Elastic Stability*, Handbuch der Physik Vol. VIa/3, ed. by S. Flügge, Springer, Berlin (1973).

[10] Liu, I-Shih: Method of Lagrange multipliers for exploitation of the entropy principle, *Arch. Rational Mech. Anal.*, 46, 131–148 (1972).

[11] Liu, I-Shih: On the requirement that material symmetries shall preserve density, *Arch. Rational Mech. Anal.* **68**, 19-26 (1978).

[12] Liu, I-Shih: On representations of anisotropic invariants, *Int. J. Engng Sci.* **20**, 1099-1109 (1982).

[13] Liu, I-Shih: On entropy flux-heat flux relation in thermodynamics with Lagrange multipliers, *Continuum Mech. thermody.* **8**, 247-256 (1996).

[14] Liu, I-Shih: *Continuum Mechanics*, Springer-Verlag, Berlin-Heidelberg (2002).

[15] Müller, I.: On the entropy inequality, *Arch. Rational Mech. Anal.* **26**, 118-141 (1967).

[16] Müller, I.: The coldness, a universal function in thermoelastic bodies, *Arch. Rational Mech. Anal.* **41**, 319-332 (1971).

[17] Noll, W.: Proof of the maximality of the orthogonal group in the unimodular group, *Arch. Rational Mech. Anal.* **18**, 100-102 (1965).

[18] Noll, W.: A mathematical theory of the mechanical behavior of continuous media, *Arch. Rational Mech. Anal.* **2**, 197-226 (1968).

[19] Smith, G. F.: On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors, *Int. J. Engng Sci.* **9**, 899-916 (1971).

[20] Treloar, L. R. G.: *The Physics of Rubber Elasticity*, Clarendon Press, Oxford (1975).

[21] Truesdell, C.: *A First Course in Rational Continuum Mechanics*, Vol. 1, Second Edition, Academic Press, Boston (1991).

[22] Truesdell, C., Noll, W.: *The Non-Linear Field Theories of Mechanics*, Handbuch der Physik, Ed. by S. Flügge, Vol. III/3, Springer (1965).

[23] Wang, C.-C.: Stress relaxation and the principle of fading memory, *Arch. Rational Mech. Anal.* **18**, 117-126 (1965).

[24] Wang, C.-C.: A new representation theorem for isotropic functions, Part I and II, *Arch. Rational Mech. Anal.* **36**, 166-197, 198-223 (1970); Corrigendum, *ibid.* **43**, 392-395 (1971).

[25] Wang, C.-C.: *Mathematical Principles of Mechanics and Electromagnetism*, Part A: Analytical and Continuum Mechanics, Plenum Press, New York and London (1979).

[26] Wang, C.-C., Truesdell, C.: *Introduction to Rational Elasticity*, Noordhoff International Publishing, Leyden (1973).

Index