

On Fourier's law of heat conduction

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A linear theory of fluid is considered in which the gradients of density, internal energy and velocity are among the constitutive variables. Thus the heat flux may be a linear combination of the gradients of density and internal energy. It is proved that this linear combination may be written as the gradient of temperature so that Fourier's law of heat conduction holds.

1 Introduction

In a systematic development of thermodynamics the absolute temperature ought to be a derived concept, as it is in classical thermostatics. Indeed, the temperature does not occur in the balance equations of mechanics and thermodynamics. Therefore it seems natural to consider as basic fields the density of mass, the velocity and the specific internal energy which all occur in the balance equations. That is done in this paper.

More specifically we consider a linear first gradient theory, in which the constitutive quantities at one point and time depend on density, velocity and internal energy at that point and time and on the values of these fields in the immediate neighborhood. We show that the heat flux can only depend on a particular combination of the gradients of density and internal energy, *viz.* the gradient of temperature.

2 Thermodynamic processes

The field equations are based on the balance laws of mass, momentum and internal energy:

$$\begin{aligned}
 \dot{\rho} + \rho v_{k,k} &= 0, \\
 \rho \dot{v}_i - T_{ik,k} &= 0, \\
 \rho \dot{e} + q_{k,k} - T_{ki} v_{i,k} &= 0.
 \end{aligned}
 \tag{1}$$

For fluids, we shall consider the density ρ , the velocity v_i and the specific internal energy ε as the thermodynamic field. The symmetric stress tensor T_{ik} and the heat flux q_k are to be given by constitutive equations of the form

$$\mathcal{C} = \hat{\mathcal{C}}(\rho, v_m, \varepsilon, \rho_{,n}, v_{m,n}, \varepsilon_{,n}). \quad (2)$$

Any field (ρ, v_i, ε) that satisfies the balance laws (1) together with constitutive equations of the form (2) is called a *thermodynamic process* in a *viscous heat-conducting fluid*.

Constitutive equations, which characterize the material behavior, cannot be entirely arbitrary. They are restricted by universal physical principles and, in particular the principle of material objectivity and the entropy principle.

3 Material objectivity and entropy principle

The principle of material objectivity requires that the constitutive function of an objective constitutive quantity must be independent of the observer. This requirement implies that: (i) the constitutive function is independent of the velocity v_i , and depends on the velocity gradient only through its symmetric part d_{ik} ; (ii) the constitutive function is an isotropic function. Therefore, we may write

$$\begin{aligned} T_{ik} &= \hat{T}_{ik}(\rho, \varepsilon, \rho_{,m}, \varepsilon_{,m}, d_{mn}), \\ q_i &= \hat{q}_i(\rho, \varepsilon, \rho_{,m}, \varepsilon_{,m}, d_{mn}), \end{aligned} \quad (3)$$

where \hat{T}_{ik} and \hat{q}_i are isotropic tensor-valued and vector-valued functions respectively.

The entropy principle will impose further restrictions on the constitutive functions. It states that for every thermodynamic process the entropy inequality must hold

$$\rho \dot{\eta} + \Phi_{k,k} \geq 0, \quad (4)$$

where the specific entropy density η and the entropy flux Φ_i are also given by constitutive equations of the form (3), *viz.*

$$\begin{aligned} \eta &= \hat{\eta}(\rho, \varepsilon, \rho_{,m}, \varepsilon_{,m}, d_{mn}), \\ \Phi_i &= \hat{\Phi}_i(\rho, \varepsilon, \rho_{,m}, \varepsilon_{,m}, d_{mn}). \end{aligned} \quad (5)$$

This constraint on the thermodynamic process can be taken into account by use of Lagrange multipliers proposed by Liu [1]: The inequality

$$\rho \dot{\eta} + \Phi_{k,k} - \Lambda^q(\dot{\rho} + \rho d_{kk}) - \Lambda^{v_i}(\rho \dot{v}_i - T_{ik,k}) - \Lambda^\varepsilon(\rho \dot{\varepsilon} + q_{k,k} - T_{ki} d_{ik}) \geq 0 \quad (6)$$

must hold for all fields $\{\rho(x_m, t), v_i(x_m, t), \varepsilon(x_m, t)\}$. The Λ^x are called *Lagrange multipliers* which are functions of $(\rho, \varepsilon, \rho_{,m}, \varepsilon_{,m}, d_{mn})$ in general.

After introducing (3) and (5) into (6), we obtain an inequality which is linear in $\dot{\rho}, \dot{v}_i, \dot{\varepsilon}, \dot{\rho}_{,m}, \dot{\varepsilon}_{,m}, \dot{d}_{mn}$ and $\rho_{,mk}, \varepsilon_{,mk}, d_{mn,k}$. Since it must hold for all values of these quantities, the coefficients of these derivatives must vanish, which leads to the

equations

$$\eta = \hat{\eta}(\varrho, \varepsilon),$$

$$\Lambda^e = \frac{\partial \hat{\eta}}{\partial \varepsilon}, \quad \Lambda^e = \varrho \frac{\partial \hat{\eta}}{\partial \varrho}, \quad \Lambda^{vi} = 0, \quad (7)$$

and

$$\frac{\partial \hat{\Phi}_{(i}}{\partial \varrho_{,m)} } = \Lambda^e \frac{\partial \hat{q}_{(i}}{\partial \varrho_{,m)} }, \quad *$$

$$\frac{\partial \hat{\Phi}_{(i}}{\partial \varepsilon_{,m)} } = \Lambda^e \frac{\partial \hat{q}_{(i}}{\partial \varepsilon_{,m)} }, \quad * \quad (8)$$

$$\frac{\partial \hat{\Phi}_i}{\partial d_{mn}} = \Lambda^e \frac{\partial \hat{q}_i}{\partial d_{mn}}.$$

There remains the residual inequality

$$\left(\frac{\partial \hat{\Phi}_k}{\partial \varrho} - \Lambda^e \frac{\partial \hat{q}_k}{\partial \varrho} \right) \varrho_{,k} + \left(\frac{\partial \hat{\Phi}_k}{\partial \varepsilon} - \Lambda^e \frac{\partial \hat{q}_k}{\partial \varepsilon} \right) \varepsilon_{,k} - \varrho \Lambda^e d_{kk} + \Lambda^e T_{ik} d_{ki} \geq 0. \quad (9)$$

Thus we conclude that the Lagrange multipliers $\Lambda^{vi} = 0$, and Λ^e, Λ^e are functions of (ϱ, ε) only, and they satisfy the relation

$$d\eta = \Lambda^e d\varepsilon + \frac{\Lambda^e}{\varrho} d\varrho. \quad (10)$$

4 Temperature and Fourier's law

To simplify further evaluation of (8) through (10), we shall introduce linear isotropic representations for constitutive equations, *viz.*

$$T_{ik} = -p \delta_{ik} + \nu d_{nn} \delta_{ik} + 2\mu d_{ik},$$

$$q_k = k_1 \varrho_{,k} + k_2 \varepsilon_{,k}, \quad (11)$$

$$\Phi_k = \phi_1 \varrho_{,k} + \phi_2 \varepsilon_{,k},$$

where the pressure p , the viscosities ν and μ as well as k_1 through ϕ_2 are all functions of (ϱ, ε) .

By use of (11), the first two relations of (8) imply immediately

$$\Phi_k = \Lambda^e q_{k}, \quad (12)$$

* Round parentheses indicate symmetrization of indices

and consequently the inequality (9) becomes

$$\nu \Lambda^\varepsilon d_{kk}^2 + 2\mu \Lambda^\varepsilon d_{ik} d_{ik} + (k_1 \varrho_{,k} + k_2 \varepsilon_{,k}) \left(\frac{\partial \Lambda^\varepsilon}{\partial \varrho} \varrho_{,k} + \frac{\partial \Lambda^\varepsilon}{\partial \varepsilon} \varepsilon_{,k} \right) - (\varrho \Lambda^\varepsilon + p \Lambda^\varepsilon) d_{kk} \geq 0. \quad (13)$$

Since this inequality has a linear term in d_{kk} , its coefficient must vanish and we conclude

$$\Lambda^\varepsilon = -\Lambda^\varepsilon \frac{p}{\varrho}. \quad (14)$$

Insertion of (14) into (10) gives

$$d\eta = \Lambda^\varepsilon \left(d\varepsilon - \frac{p}{\varrho^2} d\varrho \right) \quad (15)$$

whence — by comparison with the Gibbs equation — we conclude

$$\Lambda^\varepsilon = \frac{1}{T} \quad (16)$$

where T is the absolute temperature.

The remaining inequality becomes a positive semi-definite quadratic form, which implies that the bulk and the shear viscosities are positive,

$$\nu + \frac{2}{3} \mu \geq 0, \quad \mu \geq 0, \quad (17)$$

and that the matrix

$$-\frac{1}{T^2} \begin{bmatrix} k_1 \frac{\partial T}{\partial \varrho} & \frac{1}{2} \left(k_1 \frac{\partial T}{\partial \varepsilon} + k_2 \frac{\partial T}{\partial \varrho} \right) \\ \frac{1}{2} \left(k_1 \frac{\partial T}{\partial \varepsilon} + k_2 \frac{\partial T}{\partial \varrho} \right) & k_2 \frac{\partial T}{\partial \varepsilon} \end{bmatrix}$$

is positive-semi definite, or equivalently

$$k_1 \frac{\partial T}{\partial \varrho} \leq 0, \quad k_2 \frac{\partial T}{\partial \varepsilon} \leq 0, \quad -\frac{1}{4} \left(k_1 \frac{\partial T}{\partial \varepsilon} - k_2 \frac{\partial T}{\partial \varrho} \right)^2 \geq 0. \quad (18)$$

The last relation of (18) implies that

$$k_1 \frac{\partial T}{\partial \varepsilon} = k_2 \frac{\partial T}{\partial \varrho}, \quad (19)$$

or, if we regard ε as a function of (ϱ, T) , it becomes

$$k_1 = -k_2 \frac{\partial \varepsilon}{\partial \varrho}. \quad (20)$$

From (11), by the use of (20) the heat flux q_i becomes

$$q_i = k_2 \left(\frac{\partial \varepsilon}{\partial x_i} - \frac{\partial \varepsilon}{\partial Q} \frac{\partial Q}{\partial x_i} \right) = k_2 \frac{\partial \varepsilon}{\partial T} \frac{\partial T}{\partial x_i} \quad (21)$$

or

$$q_i = -\varkappa \frac{\partial T}{\partial x_i} \quad \text{with} \quad \varkappa = -k_2 \frac{\partial \varepsilon}{\partial T} \geq 0. \quad (22)$$

The positiveness of the thermal conductivity \varkappa is a consequence of the second relation of (18).

Equation (22) is Fourier's law of heat conduction, derived here from the seemingly more general equation (11). We have thus seen that the absolute temperature plays a special role in thermostatics as well as in heat conduction — the temperature gradient alone is the driving force for the heat flux, at least in linear theory.

References

1. Liu, I-Shih: Method of Lagrange multipliers for exploitation of the entropy principle. Arch. Rational Mech. Anal. 46 (1972) 131

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