

ON REPRESENTATIONS OF ANISOTROPIC INVARIANTS

I-SHIH LIU[†]

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530,
CEP 21910, Rio de Janeiro, Brazil

Abstract—Simple relations are established between isotropic functions and anisotropic functions through some vectors or tensors which characterized the anisotropy group. The results enable us to obtain representations of anisotropic functions using the much well-known tables for representations of isotropic functions. Transverse isotropy, orthotropy and crystal classes of triclinic, monoclinic and rhombic systems are considered.

1. INTRODUCTION

REPRESENTATIONS of isotropic functions have been extensively investigated in the past decade. Results for both functional bases [1, 2] and integrity bases [3, 4] are usually given in tables convenient for use. In fact, they have become indispensable in obtaining constitutive equations for isotropic materials, since constitutive equations must satisfy a combined objectivity-symmetry condition which requires them to be isotropic functions. For anisotropic materials in many cases, similar tables for integrity bases have also been obtained [4-10]. However they are mostly for scalar-valued functions only. Although representations for vector-valued and tensor-valued functions can be obtained from these tables, the procedure is usually very tedious.

The idea of my approach is to prove some results which enable us to obtain representations for some anisotropic invariant functions using the tables for isotropic ones. The procedure, given here can employ tables for either functional bases or integrity bases to obtain the desired representations.

Let R be the reals, V be a 3-dimensional Euclidean space and $L(V)$ be the space of second order tensors on V . Let

$$D = V^m \times L(V)^n$$

and

$$\begin{aligned}\psi: D &\rightarrow R \\ h: D &\rightarrow V \\ S: D &\rightarrow L(V).\end{aligned}\tag{1.1}$$

We say that ψ , h and S are *invariant relative to the group* $G \subset O(3)$ respectively, if for any $\mathbf{v} \in V^m$, $\mathbf{A} \in L(V)^n$ and for any $Q \in G$, we have

$$\begin{aligned}\psi(Q\mathbf{v}, QAQ^T) &= \psi(\mathbf{v}, \mathbf{A}) \\ h(Q\mathbf{v}, QAQ^T) &= Qh(\mathbf{v}, \mathbf{A}) \\ S(Q\mathbf{v}, QAQ^T) &= QS(\mathbf{v}, \mathbf{A})Q^T\end{aligned}\tag{1.2}$$

where $O(3)$ is the *full orthogonal group* on V . We have used the following abbreviations:

$$\begin{aligned}\mathbf{v} &= (v_1, \dots, v_m) \\ \mathbf{A} &= (A_1, \dots, A_n)\end{aligned}\tag{1.3}$$

[†]Present address: Hermann-Föttinger-Institut, FB9-TU Berlin, 1000 Berlin 12.

and

$$\begin{aligned} Q\mathbf{v} &= (Qv_1, \dots, Qv_m) \\ sQAQ^T &= (QA_1Q^T, \dots, QA_nQ^T) \end{aligned} \quad (1.4)$$

where

$$v_i \in V, \quad A_j \in L(V).$$

The invariants are usually called *isotropic functions* if $G = O(3)$. Their representations are well-known and widely available in the literature [1-4]. Based on these representations, we shall consider representations of invariants relative to some subgroups of $O(3)$ in this paper.

2. HEMITROPIC FUNCTIONS

Let v be a vector and W be its associated skew symmetric tensor, write

$$v = \langle W \rangle. \quad (2.1)$$

If $Q \in SO(3)$, then it is easy to verify that

$$Qv = \langle QWQ^T \rangle, \quad (2.2)$$

where $SO(3)$ is the *proper orthogonal group*.

We shall call the invariants relative to $SO(3)$ the *hemitropic functions*.

It is known that one can obtain representations of hemitropic functions by replacing vectors with their associated skew symmetric tensors. More specifically, we can state the following trivial theorems.

Theorem 2.1. For any scalar-valued or tensor-valued functions $\psi(\mathbf{v}, \mathbf{A})$, define

$$\hat{\psi}(\mathbf{W}, \mathbf{A}) = \psi(\mathbf{v}, \mathbf{A}), \quad \mathbf{v} = \langle \mathbf{W} \rangle \quad (2.3)$$

Then $\psi(\mathbf{v}, \mathbf{A})$ is a hemitropic function if and only if $\hat{\psi}(\mathbf{W}, \mathbf{A})$ is an isotropic function.

Theorem 2.2. For any vector-valued function $h(\mathbf{v}, \mathbf{A})$, let H be the skew symmetric tensor-valued function, such that $h = \langle H \rangle$ and define

$$\hat{H}(\mathbf{W}, \mathbf{A}) = H(\mathbf{v}, \mathbf{A}), \quad \mathbf{v} = \langle \mathbf{W} \rangle. \quad (2.4)$$

Then $h(\mathbf{v}, \mathbf{A})$ is a vector-valued hemitropic function if and only if $\hat{H}(\mathbf{W}, \mathbf{A})$ is a skew symmetric tensor-valued isotropic function.

Based on the above theorems, one can obtain representations for any hemitropic functions. For example, from the tables given by Wang [1, 2], one can easily construct irreducible functional bases for hemitropic functions. Therefore, we do not bother to present them here, although such tables seem to have not been given explicitly elsewhere. However, the corresponding tables of integrity bases can be found in [4].

3. GENERAL CONSIDERATIONS FOR ANISOTROPIC INVARIANTS

Many anisotropic materials possess structures which can be characterized by certain directions, lines or planes, more specifically, say characterized by some unit vectors m_1, \dots, m_a , and some tensors M_1, \dots, M_b . Let g be the group which preserves these characteristics, i.e.

$$g = \{Q \in G, Q\mathbf{m} = \mathbf{m}, Q\mathbf{M}Q^T = \mathbf{M}\} \quad (3.1)$$

where we have used similar notations introduced in (1.3) and (1.4) and G is a subgroup of $O(3)$.

Obviously, not every anisotropy material can be specified by symmetry group of the type (3.1). However, many materials do, among them, transversely isotropic, orthotropic materials, and some classes of crystalline solids. We shall treat these materials in the subsequent sections.

Now we shall prove that if the symmetry group g is of the type (3.1), one can obtain representations of invariant functions relative to g in terms of invariant functions relative to G .

Lemma. Let ψ, h, S be invariants relative to g , and $R, R' \in G$. If $R'R^T \in g$, then

$$\begin{aligned} \psi(R\mathbf{v}, R\mathbf{A}R^T) &= \psi(R'\mathbf{v}, R'\mathbf{A}R'^T) \\ R^T h(R\mathbf{v}, R\mathbf{A}R^T) &= R'^T h(R'\mathbf{v}, R'\mathbf{A}R'^T) \\ R^T S(R\mathbf{v}, R\mathbf{A}R^T)R &= R'^T S(R'\mathbf{v}, R'\mathbf{A}R'^T)R'. \end{aligned} \tag{3.2}$$

Proof. The proof is easy. For vector-valued function h , we have, since $Q = R'R^T \in g$,

$$\begin{aligned} h(R'\mathbf{v}, R'\mathbf{A}R'^T) &= h(R'R^T R\mathbf{v}, R'R^T R\mathbf{A}R^T R R'^T) \\ &= h(QR\mathbf{v}, QR\mathbf{A}R^T Q^T) \\ &= Qh(R\mathbf{v}, R\mathbf{A}R^T) \end{aligned}$$

which gives (3.2)₂. The proof for scalar-valued and tensor-valued functions are similar. ■

Let $M = \{(Q\mathbf{m}, Q\mathbf{M}Q^T), \forall Q \in G\}$ and suppose that ψ, h and S are invariants relative to g , then we can define on $D \times M$ the following functions $\hat{\psi}, \hat{h}$ and \hat{S} ,

$$\begin{aligned} \hat{\psi}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P}) &= \psi(R\mathbf{v}, R\mathbf{A}R^T) \\ \hat{h}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P}) &= R^T h(R\mathbf{v}, R\mathbf{A}R^T) \\ \hat{S}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P}) &= R^T S(R\mathbf{v}, R\mathbf{A}R^T)R \end{aligned} \tag{3.3}$$

for any $(\mathbf{v}, \mathbf{A}) \in D, (\mathbf{p}, \mathbf{P}) \in M$ and where $R \in G$, is such that

$$R\mathbf{p} = \mathbf{m}, \quad R\mathbf{P}R^T = \mathbf{M}. \tag{3.4}$$

Clearly R is not uniquely determined by the condition (3.4) in general. However, if $R' \in G$ and satisfies

$$R'\mathbf{p} = \mathbf{m}, \quad R'\mathbf{P}R'^T = \mathbf{M}, \tag{3.5}$$

then it follows that $R'R^T \in g$, since

$$\begin{aligned} R'R^T \mathbf{m} &= R'(R^T \mathbf{m}) = R'\mathbf{p} = \mathbf{m}, \\ R'R^T \mathbf{M}R R'^T &= R'(R^T \mathbf{M}R)R'^T = R'\mathbf{P}R'^T = \mathbf{M}. \end{aligned} \tag{3.6}$$

Therefore, the lemma justifies the above definition (3.3).

We have the following representation theorem:

Theorem 3.1. A function f is invariant relative to g if and only if it can be represent by

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, \mathbf{m}, \mathbf{M}) \tag{3.7}$$

where \hat{f} is invariant relative to G .

In the above theorem, the function f stands for either ψ, h or S , i.e. it is either scalar-valued, vector-valued or tensor-valued.

Proof. We shall prove for vector-valued function only. The proof for scalar-valued and tensor-valued functions are similar.

To prove the necessity, since (3.7) follows directly from the definition (3.3), we need only to show that \hat{h} is invariant relative to G . i.e.

$$\hat{h}(Q\mathbf{v}, Q\mathbf{A}Q^T, Q\mathbf{p}, Q\mathbf{P}Q^T) = Q\hat{h}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P}), \quad \forall Q \in G. \tag{3.8}$$

For any $Q \in G$, let $R \in G$ be such that

$$R(Q\mathbf{p}) = \mathbf{m}, \quad R(Q\mathbf{P}Q^T)R^T = \mathbf{M} \quad (3.9)$$

then we have by definition

$$\hat{h}(Q\mathbf{v}, QAQ^T, Q\mathbf{p}, Q\mathbf{P}Q^T) = R^T h(RQ\mathbf{v}, RQAQ^T R^T). \quad (3.10)$$

Let

$$R' = RQ \quad (3.11)$$

Then (3.9) implies that

$$R'\mathbf{p} = \mathbf{m}, \quad R'\mathbf{P}R'^T = \mathbf{M}$$

and hence, by (3.3)₂

$$h(R'\mathbf{v}, R'\mathbf{A}R'^T) = R'\hat{h}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P}).$$

Therefore (3.10) becomes

$$\hat{h}(Q\mathbf{v}, QAQ^T, Q\mathbf{p}, Q\mathbf{P}Q^T) = R^T R' \hat{h}(\mathbf{v}, \mathbf{A}, \mathbf{p}, \mathbf{P})$$

which proves (3.8) by the use of (3.11).

To prove the sufficiency, let $Q \in g$. Since $g \subset G$, and \hat{h} is invariant relative to G , (3.7) implies that

$$\begin{aligned} h(Q\mathbf{v}, QAQ^T) &= \hat{h}(Q\mathbf{v}, QAQ^T, \mathbf{m}, \mathbf{M}) \\ &= \hat{h}(Q\mathbf{v}, QAQ^T, QQ^T\mathbf{m}, QQ^T\mathbf{M}QQ^T) \\ &= Q\hat{h}(\mathbf{v}, \mathbf{A}, Q^T\mathbf{m}, Q^T\mathbf{M}Q). \end{aligned} \quad (3.12)$$

Since $Q \in g$, we have

$$Q\mathbf{m} = \mathbf{m}, \quad Q\mathbf{M}Q^T = \mathbf{M}.$$

Therefore

$$\hat{h}(\mathbf{v}, \mathbf{A}, Q^T\mathbf{m}, Q^T\mathbf{M}Q) = \hat{h}(\mathbf{v}, \mathbf{A}, \mathbf{m}, \mathbf{M}) = h(\mathbf{v}, \mathbf{A})$$

which together with (3.12) show that

$$h(Q\mathbf{v}, QAQ^T) = Qh(\mathbf{v}, \mathbf{A}), \quad \forall Q \in g.$$

This completes the proof for vector-valued functions. ■

In the following sections, we shall consider several anisotropic groups which can be characterized by (3.1) with the group G being either $O(3)$ or $SO(3)$ and hence invariants can be represented as either isotropic or hemitropic functions respectively.

4. TRANSVERSELY ISOTROPIC FUNCTIONS

Transverse isotropy is characterized by a preferred direction. Its symmetry groups can be classified into the following five classes†

†These classes are equivalent to the classes defined in Section 1.4 of [4].

$$\begin{aligned}
 g_1 &= \{Q \in SO(3), Qn = n\} \\
 g_2 &= \{Q \in O(3), Qn = n\} \\
 g_3 &= \{Q \in O(3), Q \in g_1 \text{ or } -Q \in g_1\} \\
 g_4 &= \{Q \in SO(3), Qn = n \text{ or } Qn = -n\} \\
 g_5 &= \{Q \in O(3), Qn = n \text{ or } Qn = -n\}
 \end{aligned}
 \tag{4.1}$$

where the unit vector n is the preferred direction. The smallest group g_1 , in which only rotations about n are allowed, is sometimes said to characterize *rotational symmetry*. The largest group g_5 , which contains all the other classes, is *seem* to be most suitable to characterize transverse isotropy of materials with uniaxial fibred or laminated structures. However, in the literature of representation theorems for transverse isotropy [4–7] the group mostly considered is g_2 . Here we shall give representations for all the five classes.

We shall call the invariants relative to these groups the *transversely isotropic functions*.

To apply the theorem of the previous section, we have to characterize the symmetry groups in the form of (3.1). Except g_1 and g_2 which are already in the desired form, we need the following lemmas.

Lemma 4.1. *Let N be the skew-symmetric tensor associated with the unit vector n , i.e. $n = \langle N \rangle$. If $Q \in O(3)$, then Q satisfies*

$$QNQ^T = N
 \tag{4.2}$$

if and only if $Q \in g_3$.

Proof. Let $\{n_1, n_2, n_3\}$ be an orthonormal basis such that $n_1 = n$. Then we have

$$N = n_2 \otimes n_3 - n_3 \otimes n_2.
 \tag{4.3}$$

Relative to this basis, one can easily show by direct computation that (4.2) holds if and only if Q takes the following form

$$Q = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad \forall \theta.
 \tag{4.4}$$

In other words, if $Q \in SO(3)$ then $Qn = n$, and if $-Q \in SO(3)$ then $Qn = -n$. That is, we have either $Q \in g_1$ or $-Q \in g_1$. ■

Lemma 4.2. *If $Q \in SO(3)$ (respectively $O(3)$), then Q satisfies*

$$Q(n \otimes n)Q^T = n \otimes n
 \tag{4.5}$$

if and only if $Q \in g_4$ (respectively g_5).

Proof. $n \otimes n$ is a symmetric tensor and its characteristic space is the line of the vector n . Therefore, by a well-known theorem in linear algebra, Q commutes with $n \otimes n$ if and only if Q preserves the line of n , i.e. $Qn = n$ or $Qn = -n$. ■

Now, by the theorem (3.1), we have the following results:

A transversely isotropic function $f(\mathbf{v}, \mathbf{A})$ can be represented by

(i) *relative to g_1*

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, n),$$

where \hat{f} is a hemitropic function.

(ii) *relative to g_2*

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, n),$$

where \hat{f} is an isotropic function.

(iii) relative to g_3

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, N), \quad \langle N \rangle = n,$$

where \hat{f} is an isotropic function.

(iv) relative to g_4

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, n \otimes n),$$

where \hat{f} is a hemitropic function.

(v) relative to g_5

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, n \otimes n),$$

where \hat{f} is an isotropic function.

5. ORTHOTROPIC FUNCTIONS

Orthotropic symmetry is characterized by reflections on three mutually perpendicular planes. Let their unit normals be denoted by an orthonormal set $\{n_1, n_2, n_3\}$, then the *orthotropy group* can be defined by

$$g_6 = \{Q \in O(3), Qn_i = n_i \text{ or } Qn_i = -n_i, \quad i = 1, 2, 3\}. \tag{5.1}$$

In other words, $Q \in g_6$ if and only if relative to the basis $\{n_1, n_2, n_3\}$ the matrix of Q has the diagonal form

$$Q = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{5.2}$$

where the \pm signs are not related in anyway.

We shall call the invariants relative to the orthotropy group the *orthotropic functions*.

Lemma 5.1. If $Q \in O(3)$ then Q satisfies

$$\begin{aligned} Qn_1 \otimes n_1 Q^T &= n_1 \otimes n_1, \\ Qn_2 \otimes n_2 Q^T &= n_2 \otimes n_2, \end{aligned} \tag{5.3}$$

if and only if Q belongs to the orthotropy group g_6 .

Proof. Q commutes with $n_1 \otimes n_1$ and $n_2 \otimes n_2$ if and only if Q preserves lines of n_1 and n_2 , i.e.

$$Qn_i = n_i, \quad Qn_i = -n_i, \quad i = 1, 2. \tag{5.4}$$

However, (5.4) is also valid for $i = 3$ since $Q \in O(3)$. ■

Therefore, we have the following result:

An orthotropic function $f(\mathbf{v}, \mathbf{A})$ can be represented by

$$f(\mathbf{v}, \mathbf{A}) = \hat{f}(\mathbf{v}, \mathbf{A}, n_1 \otimes n_1, n_2 \otimes n_2), \tag{5.5}$$

where \hat{f} is an isotropic function.

Clearly $n_1 \otimes n_1 + n_2 \otimes n_2 + n_3 \otimes n_3 = 1$, therefore, although one can see from the proof of the lemma (5.1) that $n_3 \otimes n_3$ can be included in the variables of \hat{f} , it is a redundant variable.

6. SOME CRYSTAL CLASSES

Besides the transverse isotropy and orthotropy groups, some crystal classes can also be described by groups of the type (3.1)

$$g = \{Q \in G, Q\mathbf{m} = \mathbf{m}, Q\mathbf{M}Q^T = \mathbf{M}\}$$

for a distinguished set of unit vectors \mathbf{m} and symmetric or skew symmetric tensors \mathbf{M} .

In other words g is characterized by the set $\{\mathbf{m}, \mathbf{M}\}$ and the group $G \subset O(3)$. Therefore, let us denote g simply by

$$g = (G; \mathbf{m}, \mathbf{M}). \quad (6.1)$$

In the following we shall give a list of such groups including the transverse isotropy groups, orthotropy group and some crystal classes. This list does not mean to be exhaustive. Let $\{n_1, n_2, n_3\}$ be an orthonormal set, and N_i be the skew symmetric tensor associated with n_i , i.e. $\langle N_i \rangle = n_i$. The definitions of the groups of crystal classes can be found in Section 1.4 of [4].

(i) *Transverse isotropy* (see (4.1))

$$\begin{aligned} g_1 &= (SO(3); n_1) \\ &= (O(3); n_1, N_1) \\ g_2 &= (O(3); n_1). \\ g_3 &= (O(3); N_1) \\ g_4 &= (SO(3); n_1 \otimes n_1) \\ g_5 &= (O(3); n_1 \otimes n_1). \end{aligned}$$

(ii) *Orthotropy* (see (5.1))

$$g_6 = (O(3); n_1 \otimes n_1, n_2 \otimes n_2).$$

(iii) *Triclinic system*

Predial class

$$g_7 = (O(3); n_1, n_2, n_3).$$

Pinacoidal class

$$g_8 = (O(3); N_1, N_2).$$

(iv) *Monoclinic system*

Domatic class

$$g_9 = (O(3); n_2, n_3).$$

Sphenoidal class

$$\begin{aligned} g_{10} &= (O(3); n_1, n_2 \otimes n_2, N_1) \\ &= (SO(3); n_2 \otimes n_2, N_1). \end{aligned}$$

Prismatic class

$$g_{11} = (O(3); n_2 \otimes n_2, N_1).$$

(v) *Rhombic system*

Pyramidal class

$$g_{12} = (O(3); n_1, n_2 \otimes n_2).$$

Disphenoidal class

$$g_{13} = (SO(3); n_2 \otimes n_2, n_3 \otimes n_3).$$

Dipyramidal class

$$g_{14} = g_6 = (O(3); n_2 \otimes n_2, n_3 \otimes n_3).$$

Some of the groups are given by two different characterizations. In general such characterizations need not be unique. The proof of the above list is almost obvious from the lemmas of the two previous sections.

7. EXAMPLES AND REMARKS

The results in the previous sections enable us to obtain representations for some anisotropic invariant functions from the much well-known representation theorems for isotropic functions by simply adding a few variables which characterize the symmetry groups. In other words, a set of invariants or generators for such a function can easily be obtained from the tables for that of isotropic functions. It must be noted that the representation obtained in this manner is not necessarily irreducible because the added variables are fixed vectors or tensors.

The following examples are based on the tables of Wang [1, 2], therefore, the bases obtained are functional bases.

Example 1. Scalar invariant functions of one symmetric tensor variable A and one vector variable v relative to the transverse isotropy groups g_2 and g_5 .

(i) Relative to g_2 ; it can be represented as an isotropic function of (v, A, n) . A basis of invariant are given as follows

$$\begin{aligned}
 & \text{tr } A, \text{ tr } A^2, \text{ tr } A^3 \\
 & v \cdot v, \quad v \cdot Av, \quad \underline{v \cdot A^2v}, \\
 & \underline{n \cdot n}, \quad n \cdot An, \quad n \cdot A^2n, \\
 & n \cdot v, \quad n \cdot Av, \quad n \cdot A^2v
 \end{aligned}
 \tag{7.1}$$

It is obvious that the above list is not an irreducible set. In fact, trivially $n \cdot n = 1$, and one can show that $v \cdot A^2v$ is a redundant element (although can not be regarded as trivial[†]), therefore they can be removed.

(ii) Relative to g_5 ; it can be represented as an isotropic function of $(v, A, n \otimes n)$. A basis of invariants are given as follows

$$\begin{aligned}
 & \text{tr } A, \text{ tr } A^2, \text{ tr } A^3, \\
 & v \cdot v, \quad v \cdot Av, \quad v \cdot A^2v, \\
 & \underline{\text{tr}(n \otimes n)}, \quad \underline{\text{tr}(n \otimes n)^2}, \quad \underline{\text{tr}(n \otimes n)^3}, \\
 & v \cdot (n \otimes n)v, \quad \underline{v \cdot (n \otimes n)^2v}, \\
 & \text{tr } A(n \otimes n), \quad \text{tr } A^2(n \otimes n), \quad \underline{\text{tr } A(n \otimes n)^2}, \quad \underline{\text{tr } A^2(n \otimes n)^2} \\
 & Av \cdot (n \otimes n)v.
 \end{aligned}
 \tag{7.2}$$

It is obvious that the underlined elements in (7.2) can all be removed because $\text{tr } n \otimes n = 1$ and $(n \otimes n)^2 = n \otimes n$. Therefore, we can rewrite the basis as

$$\begin{aligned}
 & \text{tr } A, \text{ tr } A^2, \text{ tr } A^3, \\
 & v \cdot v, \quad v \cdot Av, \quad v \cdot A^2v, \\
 & (v \cdot n)^2, \quad n \cdot An, \quad n \cdot A^2n, \\
 & (v \cdot n)(n \cdot Av).
 \end{aligned}
 \tag{7.3}$$

From this example, we note that there are some trivial redundant elements which can be

[†]Since $v \cdot A^2v = \text{tr } A^2(v \otimes v)$, and it can be reduced relative to the rest of the elements in (7.1) to essentially 2-dimensional case. Moreover, it is known that in 2-dimensional case, the trace of the product of three symmetric tensors is reducible to the traces of lower products (see [4], Section 3.2).

removed by inspection, however removing all such trivial redundant elements still does not necessarily render irreducibility of such representations due to the constancy of the added variables. We shall make no attempt for irreducibility in general in this paper.

Example 2. Invariant functions of one vector variable v relative to the transverse isotropy and orthotropy groups.

(i) g_1 —represented as hemitropic functions of (v, n)

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$v \cdot v, v \cdot n$
vector	$v, n, v \times n$
sym. tensor	$1, v \otimes v, n \otimes n, v \otimes n + n \otimes v,$ $v \otimes (v \times n) + (v \times n) \otimes v, n \otimes (n \times v) + (n \times v) \otimes n.$

(ii) g_2 —represented as isotropic functions of (v, n)

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$v \cdot v, v \cdot n$
vector	v, n
sym. tensor	$1, v \otimes v, n \otimes n, v \otimes n + n \otimes v.$

(iii) g_3 —represented as isotropic functions of $(v, N), n = \langle N \rangle$

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$v \cdot v, (v \cdot n)^2$
vector	$v, v \times n, (v \cdot n)n$
sym. tensor	$1, v \otimes v, n \otimes n, (v \times n) \otimes (v \times n),$ $v \otimes (v \times n) + (v \times n) \otimes v,$ $(v \cdot n)(n \otimes (n \times v) + (n \times v) \otimes n).$

(iv) g_4 —represented as hemitropic function of $(v, n \otimes n)$

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$v \cdot v, (v \cdot n)^2$
vector	$v, (v \cdot n)n, (v \cdot n)(v \times n)$
sym. tensor	$1, v \otimes v, n \otimes n, n \otimes (n \times v) + (n \times v) \otimes n$ $(v \cdot n)(v \otimes n + n \otimes v),$ $(v \cdot n)(v \otimes (v \times n) + (v \times n) \otimes v).$

(v) g_5 —represented as isotropic function of $(v, n \otimes n)$

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$v \cdot v, (v \cdot n)^2$
vector	$v, (v \cdot n)n$
sym. tensor	$1, v \otimes v, n \otimes n, (v \cdot n)(v \otimes n + n \otimes v).$

(vi) g_6 —represented as isotropic functions of $(v, n_1 \otimes n_1, n_2 \otimes n_2)$

<i>Functions</i>	<i>Invariants or generators</i>
scalar	$(v \cdot n_1)^2, (v \cdot n_2)^2, (v \cdot n_3)^2$
vector	$(v \cdot n_1)n_1, (v \cdot n_2)n_2, (v \cdot n_3)n_3$
sym. tensor	$n_1 \otimes n_1, n_2 \otimes n_2, n_3 \otimes n_3,$ $(v \cdot n_1)(v \otimes n_1 + n_1 \otimes v),$ $(v \cdot n_2)(v \otimes n_2 + n_2 \otimes v),$ $(v \cdot n_3)(v \otimes n_3 + n_3 \otimes v).$

In the last case we have used the identities

$$1 = n_1 \otimes n_1 + n_2 \otimes n_2 + n_3 \otimes n_3,$$

$$v = (v \cdot n_1)n_1 + (v \cdot n_2)n_2 + (v \cdot n_3)n_3$$

to rewrite the elements in a symmetric manner.

In this example, the trivial redundant elements have already been removed. The invariants or generators given are in fact irreducible.

Example 3. Scalar invariant function of two vector variables u and v relative to the triclinic-pinacoidal class g_8 .

It can be represented as an isotropic scalar-valued function of (u, v, N_1, N_2) where

$$N_1 = n_2 \otimes n_3 - n_3 \otimes n_2,$$

$$N_2 = n_3 \otimes n_1 - n_1 \otimes n_3.$$

Eliminating trivial redundants and simplifying the results, we obtain a functional basis of invariants directly from [1] in terms of components of u and v as follows

$$u_1^2, u_2^2, u_3^2, u_1u_2, u_2u_3, u_3u_1,$$

$$v_1^2, v_2^2, v_3^2, v_1v_2, v_2v_3, v_3v_1,$$

$$u_1v_1, u_2v_2, u_3v_3,$$

$$\underline{u_1v_2}, \underline{u_2v_1}, u_2v_3 - u_3v_2, u_3v_1 - u_1v_3. \tag{7.4}$$

Owing to the apparent cyclic appearance of the elements in (7.4), one is tempted to replace u_1v_2 and u_2v_1 by a single element $u_1v_2 - u_2v_1$ and hence obtain a smaller basis of 18 instead of 19 elements. This is indeed possible as we can see below.

We only have to show that if $u_1v_2 - u_2v_1$ together with the un-underlined terms in (7.4) are invariants, then u_1v_2 and u_2v_1 are also invariants.

Suppose that (u, v) and (\bar{u}, \bar{v}) are equivalent, i.e. their corresponding invariants are equal. Then, in particular, we have

$$u_1v_2 - u_2v_1 = \bar{u}_1\bar{v}_2 - \bar{u}_2\bar{v}_1 \tag{7.5}$$

and

$$v_1^2 = \bar{v}_1^2, v_1v_2 = \bar{v}_1\bar{v}_2, u_1v_1 = \bar{u}_1\bar{v}_1. \tag{7.6}$$

Now suppose that

$$u_1v_2 = \bar{u}_1\bar{v}_2 + k. \tag{7.7}$$

By (7.5) we also have

$$u_2v_1 = \bar{u}_2\bar{v}_1 + k. \tag{7.8}$$

If $v_1 = 0$, then (7.8) and (7.6)₁ imply that $k = 0$. So let us assume that $v_1 \neq 0$. Multiplying (7.7) on both sides by v_1^2 and using (7.6)₁ we get

$$(u_1v_1)(v_1v_2) = (\bar{u}_1\bar{v}_1)(\bar{v}_1\bar{v}_2) + kv_1^2$$

which by (7.6) leads to $k = 0$. Therefore

$$u_1v_2 = \bar{u}_1\bar{v}_2, u_2v_1 = \bar{u}_2\bar{v}_1.$$

In other words, we have shown that both u_1v_2 and u_2v_1 are invariants.

For the same example, an irreducible integrity basis can be found in [9] (see also in [4]), where in the place of the last three elements $u_1v_2 - u_2v_1$, $u_2v_3 - u_3v_2$ and $u_3v_1 - u_1v_3$ of the above functional basis, six elements are required. They are

$$u_1v_2, u_2v_1, u_2v_3, u_3v_2, u_3v_1, u_1v_3.$$

In other words, such an irreducible integrity basis contains 21 elements.

Remark. During the course of the present study, the authors attention was called to some papers by Boelher [11] who has taken a similar approach to the representation problems for some cases of anisotropy. Since he considered essentially symmetric tensor variables only (although its generalization is obviously inferred), the additional tensors which specify the orientation of the materials do not characterize the symmetry groups properly. The present paper gives a simple proof of the main idea and a complete treatment for functions of any number of variables: vectors, symmetric tensors and skew symmetric tensors.

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