A mixture theory of porous media and some problems of poroelasticity

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Abstract Based on the theory of an elastic solid-fluid mixture and the concept of volume fraction, a theory of porous media can be formulated consistent with basic characteristics in soil mechanics, such as Darcy's law, uplift force, and the effective stress principle. Boundary value problem for different models of poroelasticity can be considered depending on the assumptions of incompressibility of solid or fluid constituents. From the consideration of acceleration waves, there are two longitudinal waves in general, except for the model with both incompressible solid and fluid constituents, which admit only one longitudinal wave as known in the literature.

In memory of Professor Krzysztof Wilmanski

1 An elastic solid-fluid mixture

For a continuum theory of mixture, all constituents are supposed to be able to occupy the same region of space simultaneously. Let $\mathscr{B}_\alpha$ denote the constituent α and κ_α be its reference configuration and denote $B_\alpha = \kappa_\alpha(\mathscr{B}_\alpha)$. The motion of $\mathscr{B}_\alpha$ is a smooth mapping,

$$\chi_\alpha : B_\alpha \times I\!R \to I\!E, \qquad x = \chi_\alpha(X_\alpha, t), \quad X_\alpha \in B_\alpha.$$

It states that for different motion of each constituent, at the instant t, there is a material point $X_\alpha \in \mathscr{B}_\alpha$ in each constituent, $X_\alpha = \kappa_\alpha(X_\alpha)$, that occupies the same spatial position x in the Euclidean space $I\!E$. The velocity and the deformation gradient of each constituent are defined as

$$v_\alpha = \frac{\partial}{\partial t}\chi_\alpha(X_\alpha, t), \qquad F_\alpha = \nabla_{X_\alpha}\chi_\alpha(X_\alpha, t).$$

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We consider a non-reacting solid-fluid mixture ($\alpha = \mathrm{s},\mathrm{f}$) with the following balance equations of the partial mass and the partial linear momentum of each constituent, and the energy equation of the mixture:

$$\begin{aligned}
&\grave{\rho}_{\mathrm{s}} + \rho_{\mathrm{s}} \operatorname{div} v_{\mathrm{s}} = 0,\\
&\grave{\rho}_{\mathrm{f}} + \rho_{\mathrm{f}} \operatorname{div} v_{\mathrm{f}} = 0,\\
&\rho_{\mathrm{s}} \grave{v}_{\mathrm{s}} - \operatorname{div} T_{\mathrm{s}} + m_{\mathrm{f}} = \rho_{\mathrm{s}} b_{\mathrm{s}},\\
&\rho_{\mathrm{f}} \grave{v}_{\mathrm{f}} - \operatorname{div} T_{\mathrm{f}} - m_{\mathrm{f}} = \rho_{\mathrm{f}} b_{\mathrm{f}},\\
&\rho \dot{\varepsilon} + \operatorname{div} q - \operatorname{tr}(T \operatorname{grad} v) = \rho r,
\end{aligned} \tag{1.1}$$

where ρ_{s}, ρ_{f} are the partial mass densities; T_{s}, T_{f} are the partial Cauchy stresses; m_{f} is the interactive force on the fluid constituent; ε, q are the internal energy density and the heat flux of the mixture. The material derivatives with respect to the constituent and the mixture have been used,

$$\grave{y}_\alpha = \frac{\partial y_\alpha}{\partial t} + (\operatorname{grad} y_\alpha) v_\alpha, \qquad \dot{y} = \frac{\partial y}{\partial t} + (\operatorname{grad} y) v.$$

To establish field equations of the basic field variables, $\{\rho_{\mathrm{s}}, \rho_{\mathrm{f}}, \chi_{\mathrm{s}}, \chi_{\mathrm{f}}, \theta\}$, constitutive equations for the quantities in the balance equations,

$$f = \{T_{\mathrm{s}}, T_{\mathrm{f}}, \varepsilon, q, m_{\mathrm{f}}\}, \tag{1.2}$$

must be specified. For an elastic solid-fluid mixture, we consider the constitutive equations of the form:

$$f = \mathscr{F}(\theta, \rho_{\mathrm{f}}, F_{\mathrm{s}}, \operatorname{grad}\theta, \operatorname{grad}\rho_{\mathrm{f}}, \operatorname{grad} F_{\mathrm{s}}, V). \tag{1.3}$$

where θ is the temperature and $V = v_{\mathrm{f}} - v_{\mathrm{s}}$ is referred to as the relative velocity.

Thermodynamic considerations of such a mixture theory has been considered by Bowen [1] in which consequences of the entropy principle have been analyzed with Coleman-Noll procedure to obtain general restrictions on the constitutive equations. These results have been confirmed from the analysis with the use of Lagrange multipliers by Liu in [8], and can be summarized in the following constitutive equations:

$$\begin{aligned}
T_{\mathrm{f}} &= \rho_{\mathrm{f}} \psi_{\mathrm{f}} I - \frac{\partial \rho \psi_{\mathrm{I}}}{\partial \rho_{\mathrm{f}}} \rho_{\mathrm{f}} I + \rho_{\mathrm{f}} \frac{\partial \psi_{\mathrm{f}}}{\partial V} \otimes V,\\
T_{\mathrm{s}} &= \rho_{\mathrm{s}} \psi_{\mathrm{s}} I + \frac{\partial \rho \psi_{\mathrm{I}}}{\partial F_{\mathrm{s}}} F_{\mathrm{s}}^T + \rho_{\mathrm{s}} \frac{\partial \psi_{\mathrm{s}}}{\partial V} \otimes V,\\
\varepsilon &= \psi_{\mathrm{I}} - \theta \frac{\partial \psi_{\mathrm{I}}}{\partial \theta} + \frac{1}{2} \frac{\rho_{\mathrm{f}} \rho_{\mathrm{s}}}{\rho^2} V \cdot V,\\
m_{\mathrm{f}}^0 &= \frac{\partial \rho_{\mathrm{s}} \psi_{\mathrm{s}}^0}{\partial \rho_{\mathrm{f}}} \operatorname{grad} \rho_{\mathrm{f}} - \frac{\partial \rho_{\mathrm{f}} \psi_{\mathrm{f}}^0}{\partial F_{\mathrm{s}}} \cdot \operatorname{grad} F_{\mathrm{s}}, \qquad q^0 = 0.
\end{aligned} \tag{1.4}$$

where 0 denotes the equilibrium value at the state with $V = 0$ and $\operatorname{grad}\theta = 0$.

These constitutive equations depend solely on the constitutive functions of the free energy,

$$\begin{aligned} &\psi_{\mathrm{f}}=\psi_{\mathrm{f}}(\theta,\rho_{\mathrm{f}},F_{\mathrm{s}},V), \qquad \psi_{\mathrm{s}}=\psi_{\mathrm{s}}(\theta,\rho_{\mathrm{f}},F_{\mathrm{s}},V),\\ &\psi_{\mathrm{I}}=\psi_{\mathrm{I}}(\theta,\rho_{\mathrm{f}},F_{\mathrm{s}}). \end{aligned} \tag{1.5}$$

Note that although the partial free energies ψ_{f} and ψ_{s} may depend on the relative velocity V, the (inner) free energy ψ_{I},

$$\rho\,\psi_{\mathrm{I}}=\rho_{\mathrm{f}}\psi_{\mathrm{f}}+\rho_{\mathrm{s}}\psi_{\mathrm{s}},$$

does not depend on V.

Moreover, from (1.4) and (1.5), the sum of partial stresses becomes

$$\begin{aligned} T_{\mathrm{I}}&=\rho\,\psi_{\mathrm{I}}I-\frac{\partial\rho\,\psi_{\mathrm{I}}}{\partial\rho_{\mathrm{f}}}\rho_{\mathrm{f}}I+\frac{\partial\rho\,\psi_{\mathrm{I}}}{\partial F_{\mathrm{s}}}F_{\mathrm{s}}^{T},\\ T&=T_{\mathrm{I}}(\theta,\rho_{\mathrm{f}},F_{\mathrm{s}})-\frac{1}{2}\,\frac{\rho_{\mathrm{s}}\rho_{\mathrm{f}}}{\rho}V\otimes V. \end{aligned} \tag{1.6}$$

Similarly, although the partial stresses T_{f} and T_{s} may depend on V, the sum of partial stress, $T_{\mathrm{I}}=T_{\mathrm{f}}+T_{\mathrm{s}}$, does not depend on V.

If we define the equilibrium partial fluid pressure as

$$p_{\mathrm{f}}=\rho_{\mathrm{f}}\Big(\frac{\partial\rho\,\psi_{\mathrm{I}}}{\partial\rho_{\mathrm{f}}}-\psi_{\mathrm{f}}^{0}\Big), \tag{1.7}$$

then the equilibrium fluid stress reduces to the pressure, $T_{\mathrm{f}}^{0}=-p_{\mathrm{f}}I$, and

$$T_{\mathrm{f}}=-p_{\mathrm{f}}I+\rho_{\mathrm{f}}(\psi_{\mathrm{f}}-\psi_{\mathrm{f}}^{0})I+\rho_{\mathrm{f}}\frac{\partial\psi_{\mathrm{f}}}{\partial V}\otimes V, \tag{1.8}$$

and from $(1.4)_4$ and (1.7), the interaction force can be written as

$$m_{\mathrm{f}}^{0}=\frac{p_{\mathrm{f}}}{\rho_{\mathrm{f}}}\,\mathrm{grad}\,\rho_{\mathrm{f}}-\rho_{\mathrm{f}}(\mathrm{grad}\,\psi_{\mathrm{f}}^{0})\Big|_{0}. \tag{1.9}$$

Note that the definition of equilibrium fluid pressure in (1.7) implies the usual relation for a pure fluid,

$$p=\rho^{2}\,\frac{\partial\psi}{\partial\rho}. \tag{1.10}$$

2 Saturated porous media

The solid-fluid mixture considered in the previous section can be regarded as a model for saturated porous media provided that the concept of porosity is introduced. For mixture theory of porous media, a material point is regarded as a representative volume element dV which contains pores through them fluid constituent can flow. Physically, it is assumed that a representative volume element is large enough compare to solid grains (connected or not), yet at the same time small enough compare to the characteristic length of the material body.

Let the volume fraction of pores be denoted by ϕ, then for a saturated porous medium, the volume fractions of the fluid and the solid are

$$dV_{\mathrm{f}} = \phi\, dV, \qquad dV_{\mathrm{s}} = (1-\phi) dV.$$

Remember that in the mixture theory, the mass densities are defined relative to the mixture volume, so that the fluid and solid mass in the representative volume element are given by

$$dM_{\mathrm{f}} = \rho_{\mathrm{f}}\, dV = d_{\mathrm{f}}\, dV_{\mathrm{f}}, \qquad dM_{\mathrm{s}} = \rho_{\mathrm{s}}\, dV = d_{\mathrm{s}}\, dV_{\mathrm{s}},$$

and hence,

$$\rho_{\mathrm{f}} = \phi\, d_{\mathrm{f}}, \qquad \rho_{\mathrm{s}} = (1-\phi) d_{\mathrm{s}}, \tag{2.1}$$

where d_{f} and d_{s} are the true mass densities of fluid and solid constituents respectively.

2.1 Pore fluid pressure

We shall also regard the partial fluid pressure p_{f} in the mixture theory as the outcome of a "microscopic" pressure acting over the area fraction of surface actually occupied by the fluid in the pore, i.e.,

$$p_{\mathrm{f}}\, dA = P\, dA_{\mathrm{f}}, \quad \text{hence}, \quad p_{\mathrm{f}} = \phi_a P,$$

where P will be called the *pore fluid pressure* and $\phi_a = dA_{\mathrm{f}}/dA$ is the area fraction of the pores.

In general, the volume fraction ϕ and the area fraction ϕ_a may be different, yet for practical applications, we shall adopt a reasonable assumption that they are the same for simplicity, so that the *pore fluid pressure* is defined as

$$P = \frac{p_{\mathrm{f}}}{\phi}. \tag{2.2}$$

The pore pressure is an important concept in soil mechanics [3, 7].

Let us write the stresses in the following form,

$$\begin{aligned} T_{\mathrm{f}} &= -\phi P I + \overline{T}_{\mathrm{f}}, \\ T_{\mathrm{s}} &= -(1-\phi) P I + \overline{T}_{\mathrm{s}}. \end{aligned} \tag{2.3}$$

We call $\overline{T}_{\mathrm{f}}$ the *extra fluid stress* and $\overline{T}_{\mathrm{s}}$ the *effective solid stress*.

2.2 Equations of motion

The equations of motion $(1.1)_{3,4}$ for the fluid and the solid constituents can now be written as

$$
\begin{aligned}
\phi d_{\mathrm{f}} \grave{v}_{\mathrm{f}} &= -\phi \operatorname{grad} P - P \operatorname{grad} \phi + \operatorname{div} \overline{T}_{\mathrm{f}} + m_{\mathrm{f}} + \phi d_{\mathrm{f}} g, \\
(1-\phi) d_{\mathrm{s}} \grave{v}_{\mathrm{s}} &= -(1-\phi) \operatorname{grad} P + P \operatorname{grad} \phi + \operatorname{div} \overline{T}_{\mathrm{s}} - m_{\mathrm{f}} + (1-\phi) d_{\mathrm{s}} g,
\end{aligned}
\tag{2.4}
$$

where the body force is the gravitational force g.

On the other hand, from (1.9), the interactive force m_{f} in equilibrium becomes

$$
m_{\mathrm{f}}^0 = P \operatorname{grad} \phi - \phi r^0, \qquad r^0 = -\frac{P}{d_{\mathrm{f}}} \operatorname{grad} d_{\mathrm{f}} + d_{\mathrm{f}} (\operatorname{grad} \psi_{\mathrm{f}}^0)\Big|_0. \tag{2.5}
$$

By canceling out the term $P \operatorname{grad} \phi$ in (2.4) from the interactive force (2.5) leads to the following equations of motion for porous media,

$$
\begin{aligned}
\phi d_{\mathrm{f}} \grave{v}_{\mathrm{f}} &= -\phi \operatorname{grad} P + \operatorname{div} \overline{T}_{\mathrm{f}} + (m_{\mathrm{f}} - m_{\mathrm{f}}^0) - \phi r^0 + \phi d_{\mathrm{f}} g, \\
(1-\phi) d_{\mathrm{s}} \grave{v}_{\mathrm{s}} &= -(1-\phi) \operatorname{grad} P + \operatorname{div} \overline{T}_{\mathrm{s}} - (m_{\mathrm{f}} - m_{\mathrm{f}}^0) + \phi r^0 + (1-\phi) d_{\mathrm{s}} g,
\end{aligned}
\tag{2.6}
$$

2.3 Linear theory

Since equilibrium is characterized by the conditions, $\operatorname{grad} \theta = 0$ and $V = 0$, in a linear theory, we shall assume that $|\operatorname{grad} \theta|$ and $|V|$ are small quantities, and that $o(2)$ stands for higher order terms in these quantities.

From (1.8) and (2.3), the extra fluid stress,

$$
\overline{T}_{\mathrm{f}} = \phi d_{\mathrm{f}} (\psi_{\mathrm{f}} - \psi_{\mathrm{f}}^0) I + \phi d_{\mathrm{f}} \frac{\partial \psi_{\mathrm{f}}}{\partial V} \otimes V \approx o(2), \tag{2.7}
$$

is a second order quantity because the free energy of fluid constituent must be a scalar-valued isotropic function of the vector variable $(V \cdot V)$.

Moreover, we can define the resistive force as

$$
r = r^0 - \frac{1}{\phi} (m_{\mathrm{f}} - m_{\mathrm{f}}^0).
$$

It is the force against the flow of the fluid through the medium. Since the non-equilibrium part of the interactive force, $(m_{\mathrm{f}} - m_{\mathrm{f}}^0)$, vanishes in equilibrium, we can represent the resistive force as

$$
r = RV + G \operatorname{grad} \theta + r^0 + o(2). \tag{2.8}
$$

The parameter R is called the *resistivity tensor*, and its inverse R^{-1} is called the *permeability tensor*.

2.3.1 Darcy's law, uplift, and effective stress principle

The equations of motion (2.6) in the linear theory becomes

$$\begin{aligned} d_{\mathrm{f}}\grave{v}_{\mathrm{f}} &= -\operatorname{grad} P - r + d_{\mathrm{f}} g, \\ (1-\phi) d_{\mathrm{s}} \grave{v}_{\mathrm{s}} &= -(1-\phi)\operatorname{grad} P + \operatorname{div} \overline{T}_{\mathrm{s}} + \phi\, r + (1-\phi) d_{\mathrm{s}} g. \end{aligned} \tag{2.9}$$

The equation $(2.9)_1$ for the motion of the fluid is a generalized Darcy's law. Indeed, for stationary case, and only $r = RV$ is taken into account from (2.8), it reduces to the classical Darcy's law,

$$v_{\mathrm{f}} - v_{\mathrm{s}} = -R^{-1}(\operatorname{grad} P - d_{\mathrm{f}} g).$$

We can obtain an interesting equation for the solid constituent if we multiply the equation $(2.9)_1$ by $(1-\phi)$ and subtract it from the equation $(2.9)_2$,

$$(1-\phi) d_{\mathrm{s}} \grave{v}_{\mathrm{s}} - \operatorname{div} \overline{T}_{\mathrm{s}} = r + (1-\phi)(d_{\mathrm{s}} - d_{\mathrm{f}}) g + (1-\phi) d_{\mathrm{f}} \grave{v}_{\mathrm{f}}. \tag{2.10}$$

From this equation, we notice that the effective stress is not affected by the pore fluid pressure — this is known as the *effective stress principle* in soil mechanics (see [3]).

Note that there are three terms of forces on the right-hand side of the equation (2.10). The first one is the usual resistive force r of diffusive motion. The second term, $(1-\phi)(d_{\mathrm{s}} - d_{\mathrm{f}})g$, is the weight of solid reduced by the uplift (or buoyancy) from the fluid corresponding to the principle of Archimedes. The importance of uplift in soil structures had been one of the major concern in the development of soil mechanics (see [2]).

The third term, $(1-\phi) d_{\mathrm{f}} \grave{v}_{\mathrm{f}}$, is the inertia force against the displacement of fluid in the motion of the solid through it. In the theory of Biot (see [11]), the relative acceleration was introduced as a part of interactive force between solid and fluid constituents to account for the apparent added mass effect he expected in the diffusive processes. The inertia force considered here seems to correspond to such an effect. However, from the derivation above, it is clear that it is not a part of interactive force, since there is no inertia effect on the motion of fluid $(2.9)_1$.

3 Problems in poroelasticity

Hereafter we shall restrict our attention to mechanical problems (isothermal case) of the theory of elastic porous media, also known as poroelasticity. The governing equations are based on the balance equations of partial mass $(1.1)_{1,2}$ and partial momentum of fluid and solid constituents (2.9), :

$$\begin{cases} (\phi d_{\mathrm{f}})\grave{} + \phi d_{\mathrm{f}} \operatorname{div} v_{\mathrm{f}} = 0, \\ ((1-\phi) d_{\mathrm{s}})\grave{} + (1-\phi) d_{\mathrm{s}} \operatorname{div} v_{\mathrm{s}} = 0, \\ \phi\, d_{\mathrm{f}} \grave{v}_{\mathrm{f}} + \phi \ \operatorname{grad} P - \operatorname{div} \overline{T}_{\mathrm{f}} + \phi\, r = \phi\, d_{\mathrm{f}} g, \\ (1-\phi) d_{\mathrm{s}} \grave{v}_{\mathrm{s}} + (1-\phi) \operatorname{grad} P - \operatorname{div} \overline{T}_{\mathrm{s}} - \phi\, r = (1-\phi) d_{\mathrm{s}} g. \end{cases} \tag{3.1}$$

For this system of equations, from (2.3), (2.5), (2.7), and (2.8), we have the following constitutive relations:

$$\begin{aligned} &\overline{T}_{\mathrm{f}} = \phi d_{\mathrm{f}} (\psi_{\mathrm{f}} - \psi_{\mathrm{f}}^0) I + \phi d_{\mathrm{f}} \frac{\partial \psi_{\mathrm{f}}}{\partial V} \otimes V \approx o(2), \\ &\overline{T}_{\mathrm{s}} = T_{\mathrm{I}} + PI - \overline{T}_{\mathrm{f}} = T_{\mathrm{I}} + PI + o(2), \\ &r = RV - \frac{P}{d_{\mathrm{f}}} \operatorname{grad} d_{\mathrm{f}} + d_{\mathrm{f}} (\operatorname{grad} \psi_{\mathrm{f}}^0) + o(2). \end{aligned} \tag{3.2}$$

where $o(2)$ stands for higher order terms in $|V|$, and

$$\psi_{\mathrm{f}} = \psi_{\mathrm{f}}(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}, V), \qquad P = P(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}), \qquad T_{\mathrm{I}} = T_{\mathrm{I}}(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}).$$

3.1 Some models of porous media

The governing system (3.1) consists of two scalar and two vector equations, while besides the two vector variables of the motions of fluid and solid constituents, there are three scalar variables, namely, the two true densities, d_{f} and d_{s}, and the porosity ϕ. Therefore, the system is under-determinate, namely, there are less number of equations than the number of independent variables.

Porosity is a microstructural variable of the porous media. To deal with this additional variable, without postulating an additional (evolution or balance) equation for porosity, as proposed in some mixture theories in the literature ([10, 12]), there remain some possibilities to formulate deterministic theories from the present theory of porous media. We may consider following models:

1. Firstly, we can regard the porosity as a constitutive quantity, given by a constitutive relation.

   Independent variables: $(d_{\mathrm{f}}, d_{\mathrm{s}}, \chi_{\mathrm{f}}, \chi_{\mathrm{s}})$.

   Constitutive variables:

$$\begin{aligned} &P = P(d_{\mathrm{f}}, F_{\mathrm{s}}), \qquad \phi = \phi(d_{\mathrm{f}}, F_{\mathrm{s}}), \qquad r = r(d_{\mathrm{f}}, F_{\mathrm{s}}, V), \\ &\overline{T}_{\mathrm{f}} = \overline{T}_{\mathrm{f}}(d_{\mathrm{f}}, F_{\mathrm{s}}, V), \qquad \overline{T}_{\mathrm{s}} = \overline{T}_{\mathrm{s}}(d_{\mathrm{f}}, F_{\mathrm{s}}, V). \end{aligned}$$

Secondly, we can make some assumption of incompressibility of solid or fluid constituent to reduce the number of scalar variables.

2. Incompressible solid constituent: constant d_{s}.

   Independent variables: $(\phi, d_{\mathrm{f}}, \chi_{\mathrm{f}}, \chi_{\mathrm{s}})$.

   Constitutive variables:

$$
\begin{aligned}
&P = P(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}), &&r = r(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}, V),\\
&\overline{T}_{\mathrm{f}} = \overline{T}_{\mathrm{f}}(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}, V), &&\overline{T}_{\mathrm{s}} = \overline{T}_{\mathrm{s}}(\phi, d_{\mathrm{f}}, F_{\mathrm{s}}, V).
\end{aligned}
$$

3. Incompressible fluid constituent: constant d_{f}.

   Independent variables: $(\phi, d_{\mathrm{s}}, \chi_{\mathrm{f}}, \chi_{\mathrm{s}})$.

   Constitutive variables:

$$
\begin{aligned}
&P = P(\phi, F_{\mathrm{s}}), &&r = r(\phi, F_{\mathrm{s}}, V),\\
&\overline{T}_{\mathrm{f}} = \overline{T}_{\mathrm{f}}(\phi, F_{\mathrm{s}}, V), &&\overline{T}_{\mathrm{s}} = \overline{T}_{\mathrm{s}}(\phi, F_{\mathrm{s}}, V).
\end{aligned}
$$

4. Incompressible porous medium: constant d_{s} and d_{f}.

   Note that even composed with incompressible constituents, the porous body is not necessarily incompressible because the porosity may vary. Moreover, we can regard the pore pressure P as an indeterminate pressure so that the system is deterministic.

   Independent variables: $(\phi, P, \chi_{\mathrm{f}}, \chi_{\mathrm{s}})$.

   Constitutive variables:

$$
\overline{T}_{\mathrm{f}} = \overline{T}_{\mathrm{f}}(\phi, F_{\mathrm{s}}, V), \qquad \overline{T}_{\mathrm{s}} = \overline{T}_{\mathrm{s}}(\phi, F_{\mathrm{s}}, V) \qquad r = r(\phi, F_{\mathrm{s}}, V).
$$

3.2 Boundary conditions

Regarding the boundary as a singular surface between the porous body and the external medium, we have the following jump conditions for the mixture as a single body,

$$
\begin{aligned}
&[\![\rho(v-u^*)]\!]\cdot n = 0,\\
&[\![\rho v\otimes(v-u^*) - T]\!]n = 0,
\end{aligned}
\tag{3.3}
$$

where u^* is the surface velocity of the boundary.

Therefore, at the boundary of a solid-fluid mixture body, we have $v_{\rm s} = u^*$ and the jump conditions (3.3) becomes,

$$
\begin{aligned}
&[\![\rho_{\rm f} V]\!] \cdot n = 0, \\
&\left[\!\!\left[v \otimes \rho_{\rm f} V - (T_{\rm f} + T_{\rm s}) + \frac{1}{2}\frac{\rho_{\rm f}\rho_{\rm s}}{\rho} V \otimes V \right]\!\!\right] n = 0,
\end{aligned}
\tag{3.4}
$$

Furthermore, the boundary of a porous body can also be regarded as a semipermeable singular surface for the fluid constituent, in other words, the fluid can flow across the boundary and the solid cannot. In a semipermeable boundary, it has been proved that the jump condition of energy is given by (see [6]),

$$
\left[\!\!\left[\mu_{\rm f} + \tfrac{1}{2} V^2 - V \cdot \frac{\partial \psi_{\rm f}}{\partial V} \right]\!\!\right] = 0,
$$

where $\mu_{\rm f} = \dfrac{\partial \rho \psi_{\rm I}}{\partial \rho_{\rm f}}$ is the fluid chemical potential. From (1.7), $p_{\rm f} = \rho_{\rm f}(\mu_{\rm f} - \phi_{\rm f}^0)$, it implies the jump condition for the pore fluid pressure in a porous body,

$$
[P] + d_{\rm f} \left[\!\!\left[\psi_{\rm f}^0 + \frac{1}{2} V^2 - V \cdot \frac{\partial \psi_{\rm f}}{\partial V} \right]\!\!\right] = 0. \tag{3.5}
$$

Based on the above jump conditions, we can formulate the boundary condition for the system of partial differential equations. For well-posedness of the problem, two boundary conditions are needed at any point on the boundary, in addition to the proper initial conditions. There are two type of boundary conditions, namely, prescription of the motion of the boundary or the force acting on the free boundary described in the following, where the subindex w denotes the corresponding prescribed value at the exterior side of the boundary.

3.2.1 Dirichlet conditions

These are displacement (velocity) boundary conditions. From $(3.4)_1$, one can prescribe the solid displacement $u_{\rm w}$ and the fluid mass flow $m_{\rm w}$,

$$
u_{\rm s} = u_{\rm w}, \qquad \phi\, d_{\rm f}(v_{\rm f} - v_{\rm s}) \cdot n = m_{\rm w},
$$

where $u_{\rm s}$ is the displacement vector of the solid constituent.

3.2.2 Neumann conditions

Traction boundary conditions must be prescribed according to the relations $(3.4)_2$ and (3.5). Provided that the fluid mass flux is small enough, one can prescribe the total surface traction t_{w},

$$Tn = (T_{\mathrm{s}} + T_{\mathrm{f}})n = t_{\mathrm{w}}.$$

If in addition, the equilibrium free energy ψ_{f}^0 is a function of d_{f} only, then the second condition implies the continuity of the pore pressure across the boundary,

$$Pn = p_{\mathrm{w}} n,$$

where p_{w} is the pressure of the adjacent fluid acting on the boundary.

3.2.3 Remarks on boundary conditions on a free boundary

Since there are two equations of motion in the systems of governing equations, for a free boundary, two traction boundary conditions are needed. However, unlike the continuity of total traction, the continuity of pore pressure has been mostly ignored in the literature, and an additional boundary condition is often postulated for the closure of the problem.

It is proposed by Rajagobal [9] a “method of splitting the total traction” into parts acting on the fluid and the solid constituents according to the proportion of volume fraction (or surface fraction more exactly). Therefore, suppose that the boundary separates the porous body and the external fluid with pressure p_w, then the method requires that

$$T_{\mathrm{f}} n \approx -p_{\mathrm{f}} n = -\frac{\rho_{\mathrm{f}}}{d_{\mathrm{f}}}\, p_{\mathrm{w}} n.$$

Since the pore fluid pressure is defined as $P = p_{\mathrm{f}}/\phi$ and $\rho_{\mathrm{f}} = \phi d_{\mathrm{f}}$, the proposed splitting method is consistent with the the continuity of pore fluid pressure at the semipermeable surface.

Another condition was suggested by Deresiewicz [4, 11]; in which an interfacial version of Darcy’s law simulates the fluid flow across the boundary,

$$\rho_{\mathrm{f}}(v_{\mathrm{f}} - v_{\mathrm{s}}) \cdot n = \alpha\Big(p_{\mathrm{f}} - \frac{\rho_{\mathrm{f}}}{d_{\mathrm{f}}} p_{\mathrm{w}}\Big),$$

where α is referred to as the interface permeability. For $\alpha = 0$ the condition reduces to $v_{\mathrm{f}} = v_{\mathrm{s}}$, i.e., the boundary is impermeable, while for $\alpha = \infty$, it reduces to the continuity of pore fluid pressure. For the value in between, the boundary is not an ideal singular surface as proposed in the usual mixture theories, instead, the interface has its physical property. To include such an effect, a more sophisticated mixture theory containing interfacial membrane must be considered. Such a theory is beyond the present consideration. However, with of the jump condition (3.5), which relates the mass flux and the pore pressure across the boundary, it seems that the postulate of an additional condition, such as Deresiewicz condition, is superfluous in the framework of the usual mixture theories of porous media.

4 Acceleration waves

It is known in a theory of binary fluid mixture, there are two longitudinal waves, usually referred to as the first and the second sound (see [10] Appendix 5B). For porous media, the existence of two longitudinal waves, referred to as P1 and P2 waves, was predicted in Biot's theory [11]. From this observation, mathematically, the existence of two longitudinal wave in a well formulated theory of two component system seems beyond doubt in general. Nevertheless, the non-existence of second longitudinal wave in incompressible porous media has been pointed out in the literature (see [11]). Therefore, we would like to consider wave propagations in the porous models formulated from the present theory under some incompressibility assumption and show that indeed, in the model with both incompressible solid and fluid constituents, the second longitudinal wave does not exist. In other models with only one incompressible constituent two longitudinal waves may exist in general.

4.1 Wave front propagation

A propagating wave can be regarded as a moving singular surface through a material region, across which some physical quantities may suffer jump discontinuity. Let

$$[A] = A^- - A^+,$$

be the jump of the quantity A, where A^- and A^+ denote its limiting value ahead and behind of the surface respectively. Let n be the unit normal vector pointing in the propagating direction and U be the normal speed of the moving singular surface in the present configuration.

According to Hadamard lemma, if $[A] = 0$, we have the following geometric and kinematic compatibility conditions:

$$[\mathrm{grad} A] = [\mathrm{grad} A \cdot n]n, \qquad \left[\!\left[\frac{\partial A}{\partial t} \right]\!\right] = -U[\mathrm{grad} A \cdot n]. \tag{4.6}$$

We shall consider a wave front propagating into a homogeneous region of porous body in equilibrium, in which $v_{\mathrm{f}} = v_{\mathrm{s}} = 0$. For a weak discontinuity wave, i.e., on the singular surface, we assume that

$$\begin{aligned} [v_{\mathrm{f}}] &= 0, \qquad [v_{\mathrm{s}}] = 0, \qquad [F_{\mathrm{s}}] = 0, \\ [d_{\mathrm{f}}] &= 0, \qquad [d_{\mathrm{s}}] = 0, \qquad [\phi] = 0, \end{aligned}$$

and let

$$a_{\mathrm{f}} = \left[\!\left[\frac{\partial v_{\mathrm{f}}}{\partial t} \right]\!\right], \quad a_{\mathrm{s}} = \left[\!\left[\frac{\partial v_{\mathrm{s}}}{\partial t} \right]\!\right], \quad \delta_{\mathrm{f}} = \left[\!\left[\frac{\partial d_{\mathrm{f}}}{\partial t} \right]\!\right], \quad \delta_{\mathrm{s}} = \left[\!\left[\frac{\partial d_{\mathrm{s}}}{\partial t} \right]\!\right], \quad \varphi = \left[\!\left[\frac{\partial \phi}{\partial t} \right]\!\right].$$

The amplitude vectors a_{f} and a_{s} are the jumps of acceleration of fluid and solid constituents and such a weak wave is called an acceleration wave.

By repeated use of the compatibility conditions (4.6), the following jump conditions at the wave front hold:

$$\begin{aligned} &[\![\operatorname{grad} v_{\mathrm{f}}]\!] = -\frac{1}{U}\, a_{\mathrm{f}} \otimes n, && [\operatorname{grad} v_{\mathrm{s}}] = -\frac{1}{U}\, a_{\mathrm{s}} \otimes n, \\ &[\![\operatorname{grad} d_{\mathrm{f}}]\!] = -\frac{1}{U}\, \delta_{\mathrm{f}}\, n, && [\operatorname{grad} d_{\mathrm{s}}] = -\frac{1}{U}\, \delta_{\mathrm{s}}\, n, \\ &[\operatorname{grad} \phi] = -\frac{1}{U}\, \varphi\, n, && [\operatorname{grad} F_{\mathrm{s}}] = \frac{1}{U^2}\, a_{\mathrm{s}} \otimes F^T n \otimes n. \end{aligned} \tag{4.7}$$

Since the relative velocity $V = v_{\mathrm{f}} - v_{\mathrm{s}} = 0$ vanishes at the wave front, we can replace the equations of motion in $(3.1)_{3,4}$ by the equations $(2.9)_1$ and (2.10) of the linear theory,

$$\begin{aligned} &d_{\mathrm{f}} \grave{v}_{\mathrm{f}} = -\operatorname{grad} P - r + d_{\mathrm{f}} g, \\ &(1-\phi)(d_{\mathrm{s}} \grave{v}_{\mathrm{s}} - d_{\mathrm{f}} \grave{v}_{\mathrm{f}}) = \operatorname{div} \overline{T}_{\mathrm{s}} + r + (1-\phi)(d_{\mathrm{s}} - d_{\mathrm{f}}) g, \end{aligned} \tag{4.8}$$

in the systems of the governing equations, without loss of generality.

4.2 Porous media with incompressible solid constituent

We consider the porous media model of incompressible solid constituent ($d_{\mathrm{s}} =$ constant) governed by the system (3.1). With the following abbreviations for partial derivatives,

$$P_A = \frac{\partial P}{\partial A}, \qquad H_A = \frac{\partial \overline{T}_{\mathrm{s}}}{\partial A},$$

and

$$r = RV - \frac{P}{d_{\mathrm{f}}} \operatorname{grad} d_{\mathrm{f}} + d_{\mathrm{f}} \operatorname{grad} \psi_{\mathrm{f}}^0 = RV + R_{d_{\mathrm{f}}} \operatorname{grad} d_{\mathrm{f}} + R_\phi \operatorname{grad} \phi + R_{F_{\mathrm{s}}} \cdot \operatorname{grad} F_{\mathrm{s}},$$

the jumps of the governing equations at the wave front become

$$\begin{aligned} &U d_{\mathrm{f}}\, \varphi + U \phi\, \delta_{\mathrm{f}} - \phi d_{\mathrm{f}} (a_{\mathrm{f}} \cdot n) = 0, \\ &U \varphi + (1-\phi)(a_{\mathrm{s}} \cdot n) = 0, \\ &U^2 d_{\mathrm{f}} a_{\mathrm{f}} - U(P_\phi + R_\phi) \varphi\, n - U(P_{d_{\mathrm{f}}} + R_{d_{\mathrm{f}}}) \delta_{\mathrm{f}} n + \Big((R_{F_{\mathrm{s}}} + P_{F_{\mathrm{s}}}) \cdot a_{\mathrm{s}} \otimes F_{\mathrm{s}}^T n \Big) n = 0, \\ &U^2 (1-\phi)(d_{\mathrm{s}} a_{\mathrm{s}} - d_{\mathrm{f}} a_{\mathrm{f}}) + U(R_\phi I + H_\phi) \varphi\, n \\ &\qquad + U(R_{d_{\mathrm{f}}} I + H_{d_{\mathrm{f}}}) \delta_{\mathrm{f}} n - \Big((I \otimes R_{F_{\mathrm{s}}} + H_{F_{\mathrm{s}}}) \cdot a_{\mathrm{s}} \otimes F_{\mathrm{s}}^T n \Big) n = 0. \end{aligned}$$

For clarity, the last two equations in (Cartesian) component forms are

$$
\begin{aligned}
&U^2 d_\mathrm{f} a_\mathrm{f}^i - U(P_\phi + R_\phi)\varphi n_i - U(P_{d_\mathrm{f}} + R_{d_\mathrm{f}})\delta_\mathrm{f} n_i + \Big((R_{F_\mathrm{s}} + P_{F_\mathrm{s}})_{ka} a_\mathrm{s}^k F_\mathrm{s}^{ja} n_j\Big) n_i = 0,\\
&U^2(1-\phi)(d_\mathrm{s} a_\mathrm{s}^i - d_\mathrm{f} a_\mathrm{f}^i) + U(R_\phi I + H_\phi)_{ij}\varphi n_j\\
&\qquad + U(R_{d_\mathrm{f}} I + H_{d_\mathrm{f}})_{ij}\delta_\mathrm{f} n_j - \Big((\delta_{ij}(R_{F_\mathrm{s}})_{ka} + (H_{F_\mathrm{s}})_{ijka}) a_\mathrm{s}^k F_\mathrm{s}^{la} n_l\Big) n_j = 0.
\end{aligned}
$$

If we define

$$
\widehat{\Pi}_{ik} = (R_{F_\mathrm{s}} + P_{F_\mathrm{s}})_{ka} F_\mathrm{s}^{ja} n_j n_i, \qquad \widehat{Q}_{ik} = (\delta_{ij}(R_{F_\mathrm{s}})_{ka} + (H_{F_\mathrm{s}})_{ijka}) F_\mathrm{s}^{la} n_l n_j, \tag{4.9}
$$

then the system can be written as

$$
\begin{aligned}
&U d_\mathrm{f}\,\varphi + U\phi\,\delta_\mathrm{f} - \phi d_\mathrm{f}(a_\mathrm{f}\cdot n) = 0,\\
&U\,\varphi + (1-\phi)(a_\mathrm{s}\cdot n) = 0,\\
&U(P_\phi + R_\phi)\varphi n + U(P_{d_\mathrm{f}} + R_{d_\mathrm{f}})\delta_\mathrm{f} n - U^2 d_\mathrm{f} a_\mathrm{f} - \widehat{\Pi}\, a_\mathrm{s} = 0,\\
&U(R_\phi I + H_\phi)\varphi n + U(R_{d_\mathrm{f}} I + H_{d_\mathrm{f}})\delta_\mathrm{f} n - U^2(1-\phi)(d_\mathrm{f} a_\mathrm{f} - d_\mathrm{s} a_\mathrm{s}) - \widehat{Q}\, a_\mathrm{s} = 0.
\end{aligned} \tag{4.10}
$$

4.2.1 Longitudinal acceleration waves

For longitudinal waves, let

$$
a_\mathrm{f} = \hat{a}_\mathrm{f} n, \qquad a_\mathrm{s} = \hat{a}_\mathrm{s} n,
$$

i.e., the amplitude vectors of accelerations of fluid and solid constituents are in the direction of the propagation direction, and let

$$
\widehat{P}_\phi = P_\phi + R_\phi, \quad \widehat{P}_{d_\mathrm{f}} = P_{d_\mathrm{f}} + R_{d_\mathrm{f}}, \quad \widehat{H}_\phi = R_\phi + n\cdot H_\phi n, \quad \widehat{H}_{d_\mathrm{f}} = R_{d_\mathrm{f}} + n\cdot H_{d_\mathrm{f}} n.
$$

The system (4.10) then can be written as

$$
\begin{bmatrix}
U d_\mathrm{f} & U\phi & -\phi d_\mathrm{f} & 0\\
U & 0 & 0 & (1-\phi)\\
U\widehat{P}_\phi & U\widehat{P}_{d_\mathrm{f}} & -U^2 d_\mathrm{f} & -n\cdot\widehat{\Pi} n\\
U\widehat{H}_\phi & U\widehat{H}_{d_\mathrm{f}} & -U^2(1-\phi)d_\mathrm{f} & U^2(1-\phi)d_\mathrm{s} - n\cdot\widehat{Q} n
\end{bmatrix}
\begin{bmatrix}\varphi\\ \delta_\mathrm{f}\\ \hat{a}_\mathrm{f}\\ \hat{a}_\mathrm{s}\end{bmatrix}
=
\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.
$$

The propagation condition requires the determinant of the coefficient matrix be vanish.

Obviously, if $U = 0$, the determinant is identically zero, as well as $\hat{a}_\mathrm{f} = \hat{a}_\mathrm{s} = 0$, which is an uninteresting case. Therefore we are looking for moving longitudinal wave with $U \neq 0$. The vanishing of the determinant can be simplified to

$$\det \begin{bmatrix} d_{\mathrm{f}} & \phi & -\phi & 0 \\ 1 & 0 & 0 & (1-\phi) \\ \widehat{P}_\phi & \widehat{P}_{d_{\mathrm{f}}} & -U^2 & -n\cdot\widehat{\Pi}n \\ \widehat{H}_\phi & \widehat{H}_{d_{\mathrm{f}}} & -U^2(1-\phi) & U^2(1-\phi)d_{\mathrm{s}} - n\cdot\widehat{Q}n \end{bmatrix} = 0, \tag{4.11}$$

which is a quadratic equation in U^2, say

$$aU^4 + bU^2 + c = 0,$$

with the coefficients given by

$$\begin{aligned} a &= \phi(1-\phi)d_{\mathrm{s}}, \\ b &= -\phi(n\cdot\widehat{Q}n) + (1-\phi)\phi(n\cdot\widehat{\Pi}n) \\ &\quad + (1-\phi)^2\phi\widehat{P}_\phi - (1-\phi)(\phi d_{\mathrm{s}} + (1-\phi)d_{\mathrm{f}})\widehat{P}_{d_{\mathrm{f}}} + (1-\phi)(d_{\mathrm{f}}\widehat{H}_{d_{\mathrm{f}}} - \phi\widehat{H}_\phi), \\ c &= \phi\widehat{P}_{d_{\mathrm{f}}}(n\cdot\widehat{Q}n) - \phi\widehat{H}_{d_{\mathrm{f}}}(n\cdot\widehat{\Pi}n) - (1-\phi)\phi(\widehat{P}_\phi\widehat{H}_{d_{\mathrm{f}}} - \widehat{P}_{d_{\mathrm{f}}}\widehat{H}_\phi). \end{aligned}$$

In case there are two positive real roots of this quadratic equation, there will be two longitudinal waves propagating in the porous body. Obviously, this possibility depends on the constitutive parameters in the above coefficients.

To be more specific, we shall consider a simple case with constitutive relations,

$$\psi_{\mathrm{f}}^0 = \psi_{\mathrm{f}}^0(d_{\mathrm{f}}), \qquad \overline{T}_{\mathrm{s}} = \overline{T}_{\mathrm{s}}(F_{\mathrm{s}}), \tag{4.12}$$

so that

$$H_\phi = 0, \qquad H_{d_{\mathrm{f}}} = 0, \qquad R_\phi = 0, \qquad R_{F_{\mathrm{s}}} = 0.$$

Moreover, if we assume the free energy $\psi_{\mathrm{f}}^0(d_{\mathrm{f}})$ is the same as that of the pure fluid in the pore and the relation (1.10) holds, i.e.,

$$P = d_{\mathrm{f}}^2\frac{\partial\psi_{\mathrm{f}}^0}{\partial d_{\mathrm{f}}}, \qquad \text{hence} \qquad P_\phi = 0, \qquad P_{F_{\mathrm{s}}} = 0,$$

and, from (3.2), we have

$$R_{d_{\mathrm{f}}} = -\Big(\frac{P}{d_{\mathrm{f}}} - d_{\mathrm{f}}\frac{\partial\psi_{\mathrm{f}}^0}{\partial d_{\mathrm{f}}}\Big) = 0.$$

In this simple case, we have, from (4.9),

$$\begin{aligned} &\widehat{P}_\phi = 0, \qquad \widehat{\Pi} = 0, \qquad \widehat{H}_\phi = 0, \qquad \widehat{H}_{d_{\mathrm{f}}} = 0, \\ &\widehat{Q}_{ik} = (H_{F_{\mathrm{s}}})_{ijka}F_{\mathrm{s}}^{la}n_l n_j = L_{ijkl}\,n_j n_l, \end{aligned} \tag{4.13}$$

where, from (2.3),

$$L_{ijkl} = \frac{\partial T_{\mathrm{s}}^{ij}}{\partial F_{\mathrm{s}}^{ka}}F_{\mathrm{s}}^{la}$$

is the elasticity tensor of the solid constituent. It is usually assumed that the elasticity tensor is strong elliptic and the compressibility is positive, i.e.,

$$P_{d_\mathrm{f}} = \frac{\partial P}{\partial d_\mathrm{f}} > 0, \qquad L_{ijkl}\, u_i v_j u_k v_l > 0 \quad \forall\, u, v \neq 0. \tag{4.14}$$

The quadratic equation of the propagation condition can then be written as

$$(U^2 - c_1^2)(U^2 - c_2^2) - \Big(\frac{d_\mathrm{f}}{d_\mathrm{s}} \frac{1-\phi}{\phi}\, c_2^2\Big) U^2 = 0, \tag{4.15}$$

where, by (4.14),

$$c_1 = \sqrt{\frac{n \cdot \widehat{Q} n}{(1-\phi) d_\mathrm{s}}}, \qquad c_2 = \sqrt{\frac{\partial P}{\partial d_\mathrm{f}}},$$

are the well-known speeds of longitudinal wave in an elastic solid and an elastic fluid respectively. In the present case, from the roots of (4.15), these two speeds are modified by the presence of the last term in the equation (4.15).

From this simple case, it seems reasonable to expect two longitudinal wave speeds, known as P1 and P2 wave, in the present theory of porous media with incompressible solid constituent, from the propagation condition (4.11) in general.

4.2.2 Transversal acceleration wave

For transversal wave, $a_\mathrm{f} \cdot n = 0$ and $a_\mathrm{s} \cdot n = 0$, from the the first three equations of (4.10), it follows that

$$\varphi = 0, \qquad \delta_\mathrm{f} = 0, \qquad d_\mathrm{f} a_\mathrm{f} = -\frac{1}{U^2} \widehat{\Pi} a_\mathrm{s},$$

and the last equation becomes

$$\big((\widehat{Q} - (1-\phi)\widehat{\Pi}) - (1-\phi) d_\mathrm{s} U^2 I\big) a_\mathrm{s} = 0 \tag{4.16}$$

Let $a_\mathrm{s} = \hat{a}_\mathrm{s} m$ where m is a unit vector normal to to the propagation direction, i.e., $m \cdot n = 0$. Then from (4.16), there is a transversal wave with propagation speed given in the following relation,

$$(1-\phi) d_\mathrm{s} U^2 = m \cdot \big(\widehat{Q} - (1-\phi)\widehat{\Pi}\big) m.$$

In particular, for the simple case given by (4.12), the velocity of propagation reduces to the well-known speed of shear wave in an elastic body,

$$U = c_s = \sqrt{\frac{m \cdot \widehat{Q} m}{(1-\phi) d_\mathrm{s}}}.$$

4.3 Porous media with incompressible fluid constituent

Similarly, for the model of incompressible fluid constituent ($d_{\mathrm{f}} = \text{constant}$) governed by the system (3.1), we have

$$
\begin{aligned}
&U\varphi - \phi(a_{\mathrm{f}}\cdot n) = 0,\\
&Ud_{\mathrm{s}}\varphi - U(1-\phi)\delta_{\mathrm{s}} + (1-\phi)d_{\mathrm{s}}(a_{\mathrm{s}}\cdot n) = 0,\\
&U\widehat{P}_\phi\varphi n - U^2 d_{\mathrm{f}} a_{\mathrm{f}} - \widehat{\Pi}\,a_{\mathrm{s}} = 0,\\
&U(R_\phi I + H_\phi)\varphi n - U^2(1-\phi)(d_{\mathrm{f}}a_{\mathrm{f}} - d_{\mathrm{s}}a_{\mathrm{s}}) - \widehat{Q}\,a_{\mathrm{s}} = 0.
\end{aligned}
\tag{4.17}
$$

For longitudinal waves, we have

$$
\begin{bmatrix}
U & 0 & -\phi & 0\\
Ud_{\mathrm{s}} & -U(1-\phi) & 0 & (1-\phi)d_{\mathrm{s}}\\
U\widehat{P}_\phi & 0 & -U^2 d_{\mathrm{f}} & -n\cdot\widehat{\Pi}n\\
U\widehat{H}_\phi & 0 & -U^2(1-\phi)d_{\mathrm{f}} & U^2(1-\phi)d_{\mathrm{s}} - n\cdot\widehat{Q}n
\end{bmatrix}
\begin{bmatrix}\varphi\\ \delta_{\mathrm{s}}\\ \hat{a}_{\mathrm{f}}\\ \hat{a}_{\mathrm{s}}\end{bmatrix}
=
\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.
$$

and the determinant of the coefficient matrix reduces to

$$
\det\begin{bmatrix}
1 & -\phi & 0\\
\widehat{P}_\phi & -U^2 d_{\mathrm{f}} & -n\cdot\widehat{\Pi}n\\
\widehat{H}_\phi & -U^2(1-\phi)d_{\mathrm{f}} & U^2(1-\phi)d_{\mathrm{s}} - n\cdot\widehat{Q}n
\end{bmatrix} = 0,
\tag{4.18}
$$

which lead to the quadratic equation in U^2,

$$
\big(U^2(1-\phi)d_{\mathrm{s}} - n\cdot\widehat{Q}n\big)\big(U^2 d_{\mathrm{f}} - \phi\widehat{P}_\phi\big) + (n\cdot\widehat{\Pi}n)\big(U^2(1-\phi)d_{\mathrm{f}} - \phi\widehat{H}_\phi\big) = 0.
$$

Therefore, there may also exist two longitudinal wave in the incompressible fluid model.

We may also consider a simple case that the pore pressure and the fluid free energy are independent of F_{s}, so that $\widehat{\Pi} = 0$. In this case, there are two longitudinal waves (P1 and P2 waves),

$$
U_1 = \sqrt{\frac{n\cdot\widehat{Q}n}{(1-\phi)d_{\mathrm{s}}}}, \qquad U_2 = \sqrt{\frac{\phi}{d_{\mathrm{f}}}\widehat{P}_\phi}.
$$

The first wave is the same as the longitudinal wave in the elastic body, while the second wave is essentially the compressive wave of pore fluid.

For transversal wave, from (4.17), one can easily check that the propagation condition is exactly the same as the relation (4.16) for the incompressible solid model.

4.4 Incompressible porous media

For an incompressible porous medium, we assume that both the solid and the fluid true densities are constant. In this model the pore pressure is regarded as an indeterminate pressure, and for acceleration waves, we have, in addition,

$$[P] = 0, \qquad [\operatorname{grad} P] = \pi n,$$

where $\pi = [\operatorname{grad} P \cdot n]$. By taking the jump of the system of equations (3.1) at the wave front, we obtain

$$\begin{aligned}
&U\varphi - \phi(a_\mathrm{f} \cdot n) = 0, \\
&U\varphi + (1-\phi)(a_\mathrm{s} \cdot n) = 0, \\
&UR_\phi \varphi n - U^2 \pi n - U^2 d_\mathrm{f} a_\mathrm{f} - \widehat{\Pi}\, a_\mathrm{s} = 0, \\
&U(R_\phi I + H_\phi)\varphi n - U^2(1-\phi)(d_\mathrm{f} a_\mathrm{f} - d_\mathrm{s} a_\mathrm{s}) - \widehat{Q}\, a_\mathrm{s} = 0.
\end{aligned} \tag{4.19}$$

The abbreviations are the same as before except

$$\widehat{\Pi}_{ik} = (R_{F_\mathrm{s}})_{ka} F_\mathrm{s}^{ja} n_j n_i.$$

For longitudinal wave, the system can be written as

$$\begin{bmatrix} U & 0 & -\phi & 0 \\ U & 0 & 0 & (1-\phi) \\ UR_\phi & -U^2\pi & -U^2 d_\mathrm{f} & -n\cdot\widehat{\Pi} n \\ U\widehat{H}_\phi & 0 & -U^2(1-\phi)d_\mathrm{f} & U^2(1-\phi)d_\mathrm{s} - n\cdot\widehat{Q}n \end{bmatrix} \begin{bmatrix} \varphi \\ \pi \\ \hat{a}_\mathrm{f} \\ \hat{a}_\mathrm{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and the determinant of the coefficient matrix reduces to

$$\det \begin{bmatrix} 1 & -\phi & 0 \\ 1 & 0 & (1-\phi) \\ \widehat{H}_\phi & -U^2(1-\phi)d_\mathrm{f} & U^2(1-\phi)d_\mathrm{s} - n\cdot\widehat{Q}n \end{bmatrix} = 0,$$

which gives only one propagation speed,

$$U^2 = \frac{\phi\, n\cdot\widehat{Q}n + \phi(1-\phi)\widehat{H}_\phi}{\phi(1-\phi)d_\mathrm{s} + (1-\phi)^2 d_\mathrm{f}}.$$

Therefore, there is no second longitudinal wave in this model as pointed out in the literature. As for the transversal wave, the propagation speed is the same as before.

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