

On pore fluid pressure and effective solid stress in the mixture theory of porous media

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Abstract In this paper we briefly review a typical example of a mixture of elastic materials, in particular, an elastic solid-fluid mixture as a model for porous media. Application of mixture theories to porous media rests upon certain physical assumptions and appropriate interpretations in order to be consistent with some better-known notions in engineering applications. We shall discuss, for instance, the porosity, pore fluid pressure, effective solid stress and Darcy's law in this paper.

Dedicated to Krzysztof Wilmański on the occasion of his 70th birthday

1 Introduction

The essential features of theories of mixtures in the framework of continuum mechanics have been developed throughout the sixties and seventies. Here we briefly review a typical example of a mixture of elastic materials, in particular, an elastic solid-fluid mixture as a model for porous media. We shall outline a simple model of porous media by introducing the volume fraction, pore fluid pressure, effective solid stress, etc., as reinterpretations of the results of mixture theory to porous media. Some elementary results in soil mechanics are obtained, in particular, a generalization of the Darcy's law from the equation of motion of the fluid constituent, and the effective stress principle. As an example of an equilibrium solution in a linear theory for incompressible porous media, porosity distribution in saturated soil is considered.

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2 Mixture of elastic materials

For a mixture of N constituents, we introduce the following quantities for the constituent $\alpha \in N$ and the mixture:

ρ_α	partial mass density of constituent α .
\mathbf{v}_α	partial velocity of constituent α .
T_α	partial stress tensor of constituent α .
\mathbf{b}_α	external body force on constituent α .
\mathbf{m}_α	interaction force on constituent α .
ρ	mass density of mixture.
\mathbf{v}	velocity of mixture.
T	stress tensor of mixture.
ε	internal energy density of mixture.
\mathbf{q}	energy flux of mixture.
r	energy supply of mixture.

Following the pioneering work of Truesdell ([5], see also [6]), the basic laws of a non-reacting mixture are given by the following balance equations for mass and linear momentum for each constituent:

$$\begin{aligned} \frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha) &= 0, \\ \frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - T_\alpha) &= \rho_\alpha \mathbf{b}_\alpha + \mathbf{m}_\alpha. \end{aligned} \quad (1)$$

and for the mixture as a whole, by defining

$$\rho = \sum_\alpha \rho_\alpha, \quad \mathbf{v} = \sum_\alpha \frac{\rho_\alpha}{\rho} \mathbf{v}_\alpha, \quad \mathbf{b} = \sum_\alpha \frac{\rho_\alpha}{\rho} \mathbf{b}_\alpha,$$

and

$$T = \sum_\alpha (T_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha),$$

where

$$\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}$$

is the diffusion velocity of constituent α , and by requiring

$$\sum_\alpha \mathbf{m}_\alpha = 0,$$

we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - T) &= \rho \mathbf{b}. \end{aligned} \quad (2)$$

which together with the energy equation,

$$\frac{\partial \rho \varepsilon}{\partial t} + \operatorname{div}(\rho \varepsilon \mathbf{v} + \mathbf{q}) - \operatorname{tr}(T \operatorname{grad} \mathbf{v}) = \rho r. \quad (3)$$

constitute the basic balance laws of the mixture as a single body.

We consider a mixture of elastic materials characterized by the constitutive equation of the form:

$$f = \mathcal{F}(\theta, \operatorname{grad} \theta, F_\alpha, \operatorname{grad} F_\alpha, \mathbf{v}_\alpha),$$

where θ is the temperature, F_α is the deformation gradient of the constituent α relative to a reference configuration, and

$$f = \{T_\alpha, \varepsilon, \mathbf{q}, \mathbf{m}_\alpha\}$$

are the constitutive quantities for the basic field variables $\{\rho_\alpha, \mathbf{v}_\alpha, \theta\}$ with governing equations consisting of (1) and (3). Constitutive theory of such a mixture has been considered by Bowen [1] in which consequences of the entropy principle have been obtained based on the entropy inequality of the form:

$$\frac{\partial \rho \eta}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v} + \Phi) - \rho s \geq 0, \quad (4)$$

where η is the entropy density of the mixture. The entropy flux and the entropy supply density are given by

$$\begin{aligned} \Phi &= \frac{1}{\theta} \left(\mathbf{q} - \sum_\alpha (\rho_\alpha (\psi_\alpha + \frac{1}{2} \mathbf{u}_\alpha^2) I - T_\alpha^T) \mathbf{u}_\alpha \right), \\ s &= \frac{1}{\theta} \left(r - \frac{1}{\rho} \sum_\alpha \rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{b}_\alpha \right), \end{aligned} \quad (5)$$

where ψ_α is the free energy density of the constituent α , and for the mixture we define

$$\psi = \frac{1}{\rho} \sum_\alpha \rho_\alpha \psi_\alpha,$$

and I is the identity tensor.

Remark

The specific form of the relations (5) can be derived provided that the Clausius-Duhem assumptions on the entropy flux and the entropy supply are valid for each constituent. However, only the entropy production of the mixture as a whole is postulated to be non-negative. It has been shown that such assumptions are appropriate (but not necessarily general enough) to account for the behavior of a mixture within the framework of continuum mechanics. On the other hand, the very expressions in (5) show that the Clausius-Duhem assumptions $\Phi = \mathbf{q}/\theta$ and $s = r/\theta$ are not general enough to account for material bodies in general.

2.1 Summary of results for elastic solid-fluid mixtures

For a mixture of a solid (with subindex s) and a fluid (with subindex f), the constitutive equations take the form:

$$f = \mathcal{F}(\theta, \text{grad } \theta, \rho_f, F_s, \text{grad } \rho_f, \text{grad } F_s, V),$$

where $V = \mathbf{v}_f - \mathbf{v}_s$ is the relative velocity of the fluid through the solid.

We shall summarize the constitutive results from thermodynamic considerations obtained by Bowen ([1], see also [6] Appendix 5A) in the case of an elastic solid-fluid mixture:

$$\begin{aligned} \psi &= \psi(\theta, \rho_f, F_s), \\ T_f &= \rho_f \psi_f I - \frac{\partial \rho \psi}{\partial \rho_f} \rho_f I + \rho_f \frac{\partial \psi_f}{\partial V} \otimes V, \\ T_s &= \rho_s \psi_s I + \frac{\partial \rho \psi}{\partial F_s} F_s^T + \rho_s \frac{\partial \psi_s}{\partial V} \otimes V. \end{aligned} \quad (6)$$

Since there are only two constituents, we have $\mathbf{m}_f + \mathbf{m}_s = 0$ and we shall write $\mathbf{m} = \mathbf{m}_f = -\mathbf{m}_s$. Moreover, we have

$$\begin{aligned} \psi_f^0 &= \widehat{\psi}_f(\theta, \rho_f, F_s), \\ \psi_s^0 &= \widehat{\psi}_s(\theta, \rho_f, F_s), \\ \mathbf{m}^0 &= \frac{\partial \rho_s \widehat{\psi}_s}{\partial \rho_f} \text{grad } \rho_f - \frac{\partial \rho_f \widehat{\psi}_f}{\partial F_s} [\text{grad } F_s], \end{aligned} \quad (7)$$

where the superscript 0 indicates the value in equilibrium defined as processes for which $\text{grad } \theta = 0$ and $V = 0$.

If we define the (equilibrium) chemical potential of the fluid and the (equilibrium) partial fluid pressure as

$$\mu_f = \frac{\partial \rho \psi}{\partial \rho_f}, \quad p_f = \rho_f (\mu_f - \psi_f^0), \quad (8)$$

then we have

$$T_f^0 = -p_f I,$$

and the relation (7)₃ can be written as

$$\mathbf{m}^0 = \frac{p_f}{\rho_f} \text{grad } \rho_f - \rho_f (\text{grad } \widehat{\psi}_f)^0. \quad (9)$$

2.2 Jump condition at fluid-permeable surface

In [2] Liu considered an ideal fluid-permeable surface, i.e., a surface across which there is no jump of temperature and the solid constituent does not go through,

$$[\theta] = 0, \quad \mathbf{v}_s = \mathbf{u}^*,$$

where $[\]$ denotes the jump across the surface and \mathbf{u}^* is the surface velocity. It is proved that at such surfaces, the following jump condition holds,

$$\left[\left[\mu_f + \frac{1}{2}(\mathbf{v}_f - \mathbf{u}^*)^2 - (\mathbf{v}_f - \mathbf{u}^*) \cdot \frac{\partial \psi_f}{\partial \mathbf{v}_f} \right] \right] = 0. \quad (10)$$

In particular, if $\mathbf{v}_f = \mathbf{u}^*$, the the chemical potential of the fluid constituent is continuous over the surface.

3 Saturated porous media

The solid-fluid mixture considered in the previous section can be regarded as a model for saturated porous media provided that the concept of porosity, the volume fraction of the fluid constituent ϕ , is introduced:

$$\rho_f = \phi d_f, \quad \rho_s = (1 - \phi) d_s, \quad (11)$$

where d_f and d_s are the true mass densities of the fluid and the solid constituents respectively.

3.1 Pore fluid pressure

We shall also regard the partial fluid pressure p_f in the mixture theory as the outcome of a “microscopic” pressure acting over the area fraction of surface actually occupied by the fluid in the pore. Thus we define the *pore fluid pressure* P as

$$P = \frac{p_f}{\phi}. \quad (12)$$

In this definition, we have tacitly assumed that the surface fraction is the same as the volume fraction ϕ defined in (11), which, of course, may not be true in general but is acceptable for most practical applications.

From (7), we have

$$\psi_f^0 = \widehat{\psi}_f(\theta, \phi d_f, F_s).$$

However, in most applications, it is reasonable to assume that in equilibrium the free energy of the fluid constituent is the same as the free energy of the pure fluid, i.e., we shall assume that

$$\psi_f^0 = \widehat{\psi}_f(\theta, d_f). \quad (13)$$

From (8), we have

$$\mu = \frac{P}{d_f} + \psi_f^0.$$

At an ideal fluid-permeable surface which allows the fluid to go through, since the true density does not change, $[d_f] = 0$, it follows that

$$[\mu] = \left[\left[\frac{P}{d_f} \right] \right] + [\widehat{\psi}_f(\theta, d_f)] = \frac{1}{d_f} [P].$$

Therefore the condition (10) implies that

$$[P] + d_f \left[\left[\frac{1}{2} (\mathbf{v}_f - \mathbf{u}^*)^2 - (\mathbf{v}_f - \mathbf{u}^*) \cdot \frac{\partial \psi_f}{\partial \mathbf{v}_f} \right] \right] = 0.$$

In particular, if $\mathbf{v}_f = \mathbf{u}^*$, then

$$[P] = 0. \quad (14)$$

This result agrees with our physical intuition that if the fluid does not flow from one side to the other side through the pores then the pressures in the pores on both sides must be equal. In [2], it has been shown by virtue of the jump condition (14) in equilibrium, that the fluid pressure measured from the manometer tube attached to the porous body is the pore fluid pressure.

3.2 Equations of motion

The equations of motion for the fluid and the solid constituents in porous media can be written as

$$\begin{aligned} \phi d_f \dot{\mathbf{v}}_f &= \operatorname{div} T_f + \mathbf{m} + \phi d_f \mathbf{g}, \\ (1 - \phi) d_s \dot{\mathbf{v}}_s &= \operatorname{div} T_s - \mathbf{m} + (1 - \phi) d_s \mathbf{g}, \end{aligned}$$

where $\dot{\mathbf{v}}_f$ and $\dot{\mathbf{v}}_s$ are the accelerations of the fluid and the solid respectively, and the external body force is the gravitational force \mathbf{g} .

Let us write the stresses in the following form,

$$\begin{aligned} T_f &= -\phi PI + \overline{T}_f, \\ T_s &= -(1 - \phi)PI + \overline{T}_s. \end{aligned} \quad (15)$$

We call \overline{T}_f the *extra fluid stress* and \overline{T}_s the *effective solid stress*, since it reduces to the effective stress widely used in soil mechanics as we shall see later.

The equations of motion then become

$$\begin{aligned}\phi d_f \dot{\mathbf{v}}_f &= -\phi \text{grad} P - P \text{grad} \phi + \text{div} \bar{T}_f + \mathbf{m} + \phi d_f \mathbf{g}, \\ (1-\phi) d_s \dot{\mathbf{v}}_s &= -(1-\phi) \text{grad} P + P \text{grad} \phi + \text{div} \bar{T}_s - \mathbf{m} + (1-\phi) d_s \mathbf{g}.\end{aligned}\quad (16)$$

We obtain from (9), the interactive force \mathbf{m} in equilibrium,

$$\mathbf{m}^0 = P \text{grad} \phi + \phi \left(\frac{P}{d_f} - d_f \frac{\partial \hat{\psi}_f}{\partial d_f} \right) \text{grad} d_f, \quad (17)$$

by the use of (12) and (13). We shall call the non-equilibrium part of the interactive force

$$\mathbf{r} = \mathbf{m} - \mathbf{m}_0 \quad (18)$$

the *resistive force*, since it is the force against the flow of the fluid through the medium. We may also call $-\mathbf{r}$ the *drag force* acting upon the solid constituent.

3.3 Linear theory and Darcy's law

If we further assume that $|\text{grad} \rho_f|$, $|\text{grad} F_s|$, $|\text{grad} \theta|$, and $|V|$ are small quantities, then from (6), we have

$$\bar{T}_f = \rho_f \frac{\partial \psi_f}{\partial V} \otimes V \approx o(2)$$

is a second order quantity because the free energy of fluid constituent must be a scalar-valued isotropic function. Moreover, since the resistive force vanishes in equilibrium, i.e., $\text{grad} \theta = 0$ and $V = 0$, we can write

$$\mathbf{r} = -\phi R V - \phi G \text{grad} \theta + o(2), \quad (19)$$

where R and G are material parameters, which are tensor-valued functions of (θ, ϕ, d_f, F_s) .

Therefore, by the use of (17) through (19), the equations of motion for the fluid constituent (16)₁ in the linear theory becomes

$$d_f \dot{\mathbf{v}}_f = -\text{grad} P - \mathbf{k} + d_f \mathbf{g}, \quad (20)$$

where

$$\mathbf{k} = R V + G \text{grad} \theta - \left(\frac{P}{d_f} - d_f \frac{\partial \hat{\psi}_f}{\partial d_f} \right) \text{grad} d_f. \quad (21)$$

The equation of motion for the fluid constituent (20) is a generalization of *Darcy's law*. It is shown in [2], that for the case of classical Darcy's experiment (see [3]), the equation reduces to the original form of the Darcy's law, where the reciprocal of the resistivity constant R is called the *permeability tensor*.

Similarly, the equation (16)₂ becomes

$$(1-\phi) d_s \dot{\mathbf{v}}_s = -(1-\phi) \text{grad} P + \text{div} \bar{T}_s + \phi \mathbf{k} + (1-\phi) d_s \mathbf{g}, \quad (22)$$

which is the equation of motion of the solid constituent.

We can obtain an interesting equation for the solid constituent if we multiply the equation (20) by $-(1 - \phi)$ and add it to the equation (22),

$$(1 - \phi)(d_s \dot{\mathbf{v}}_s - d_f \dot{\mathbf{v}}_f) = \operatorname{div} \bar{T}_s + \mathbf{k} + (1 - \phi)(d_s - d_f)\mathbf{g}. \quad (23)$$

In quasi-static case when the accelerations can be neglected, the two equations (20) and (23) for the fluid and the solid constituents can be written as

$$\begin{aligned} \operatorname{grad} P + \mathbf{k} &= d_f \mathbf{g}, \\ -\operatorname{div} \bar{T}_s - \mathbf{k} &= (1 - \phi)(d_s - d_f)\mathbf{g}. \end{aligned} \quad (24)$$

Note that the term $(1 - \phi)(d_s - d_f)\mathbf{g}$, which represents the difference between the gravitational force of the solid and the fluid in the volume fraction of the solid, can be regarded as the *buoyancy force* on the solid constituent.

Furthermore, adding the two equations in (24) together, we obtain

$$-\operatorname{div} T = \rho \mathbf{g}. \quad (25)$$

Clearly this is the equilibrium equation for the mixture as a whole, where from (15)

$$T = T_f + T_s = -PI + \bar{T}_s + o(2), \quad \rho = \phi \rho_f + (1 - \phi)\rho_s, \quad (26)$$

are the total stress and the total mass density of the mixture respectively.

4 Incompressible porous media

We shall consider incompressible porous media, i.e., for which the true mass densities d_f and d_s are constants. For a linear theory, we have

$$\begin{aligned} T_f &= -\phi PI, \\ T_s &= -(1 - \phi)PI + \bar{T}_s, \\ \mathbf{k} &= RV + G \operatorname{grad} \theta. \end{aligned} \quad (27)$$

where the pore pressure P is constitutively indeterminate and the effective solid stress is given by the constitutive relation, $\bar{T}_s = \bar{T}_s(\phi, F_s)$.

From (24), the equilibrium equations become

$$\begin{aligned} \operatorname{grad} P &= d_f \mathbf{g}, \\ -\operatorname{div} \bar{T}_s &= (1 - \phi)(d_s - d_f)\mathbf{g}. \end{aligned} \quad (28)$$

These two equations can be solved separately. Suppose that the x -coordinate is in the vertical downward direction, $\mathbf{g} = g\mathbf{e}_x$. Then the first equation can be integrated immediately to give

$$P = d_f g x + P_0, \quad P_0 = P(0). \quad (29)$$

This result asserts that the equilibrium pore pressure is the hydrostatic pressure. It agrees with the observation in soil mechanics from experimental measurements that the manometer pressure in the soil is the pressure as if the medium were bulk fluid, unaffected by the presence of the solid constituent in the medium.

We remark that this result is sometimes overlooked in the mixture theory of porous media. It is mainly due to the fact that in the theory of *simple mixture*, which omits the second gradients of deformations as independent constitutive variables, the equilibrium interactive force \mathbf{m}^0 is identically zero by constitutive hypothesis. However from the relation (17), \mathbf{m}_0 is not a negligible quantity for a body with non-uniform porosity, $\mathbf{m}_0 = P \text{grad } \phi$, and it is easy to see that in the absence of this term, the result (29) need not follow. This remark, we shall regard as a strong evidence that porous media must be treated as *non-simple mixtures* even for a linear equilibrium theory.

4.1 Effective stress principle

Now let us turn to the equation (28)₂ for the solid constituent, or equivalently the equilibrium equation for the whole mixture (25). After this equation is solved for the total stress T , the effective stress can be obtained from (26),

$$\bar{T}_s = T + PI,$$

where the pore pressure P is given by (29). This is the *effective stress principle* in soil mechanics first introduced by Terzaghi (see [4]).

4.2 An equilibrium solution

As an application of the equilibrium equation of the solid constituent to determine the porosity distribution in saturated soil, we shall consider a simple constitutive model that

$$\bar{T}_s = -\pi(\phi)I, \quad (30)$$

where π will be called the effective solid pressure. We have from (28)₂,

$$\text{grad } \pi = (1 - \phi)((d_s - d_f)\mathbf{g}.$$

First we note that if the effective pressure is constant, then either $\phi = 1$ or $d_s = d_f$. Both are uninteresting cases.

We shall further assume the constitutive relation for the effective pressure is given by a power law:

$$\pi(\phi) = b(1 - \phi)^a, \quad (31)$$

where a and b are non-zero material constants. Then we have

$$-ab(1-\phi)^{a-2} \frac{d\phi}{dx} = (d_s - d_f)g.$$

For $a \neq 1$, integration of the equation leads to

$$(1-\phi)^{a-1} = \frac{a-1}{ab}(d_s - d_f)gx + (1-\phi_0)^{a-1}, \quad (32)$$

while for $a = 1$, it gives

$$(1-\phi) = (1-\phi_0) \exp\left(\frac{1}{b}(d_s - d_f)gx\right), \quad (33)$$

where $\phi_0 = \phi(0)$.

If experimental measurements of the porosity ϕ versus x in the soil are taken, then one can compare them with the solution (32) or (33) for the determination of the material constants a and b in the power law (31). It offers a check for the applicability of the present theory based on such a simple model.

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