ENCYCLOPEDIA OF LIFE SUPPORT SYSTEMS

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Encyclopedia of Life Support Systems

6.161.6

CONSTITUTIVE THEORIES: BASIC PRINCIPLES

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Keywords: Frame of reference, change of observer, Euclidean objectivity, thermomechanical history, constitutive equation, material frame-indifference, material objectivity, simple materials, material symmetry, isotropic functions, entropy principle.

Contents

Summary

Constitutive theories deal with mathematical models of material bodies. Aside from physical experiences and experimental observations, theoretical considerations for constitutive equations rest upon some basic principles, among them the most fundamental ones being material frame-indifference and material symmetry. The former states that constitutive equations characterize intrinsic properties of a material body and thereby must be independent of whoever is describing them, while the later requires constitutive equations to comply with symmetry properties of a material body relative to change of reference state. General framework of constitutive formulation consistent with these two requirements is presented and for some classes of material bodies, including elastic solids, viscous fluids and heat conductors, general constitutive equations are analyzed and classical constitutive laws are obtained. Thermodynamic considerations are also important in constitutive theories. It is formulated in the entropy principle which requires constitutive equations to be consistent with the second law of thermodynamics. The exploitation of entropy principle to obtain constitutive restrictions is a major task in rational thermodynamics. As an example, for the class of elastic materials, Gibbs relation of classical thermostatics and Fourier inequality are obtained. Through derivations of the well-known classical results, the presentation in this chapter emphasizes mathematical implications of basic principles in rational framework of constitutive theories for general material bodies.

1 Introduction

Properties of material bodies are described mathematically by constitutive equations. Classical models, such as Hooke's law of elastic solids, Navier-Stokes law of viscous fluids, and Fourier law of heat conduction, are mostly proposed based on physical experiences and experimental observations. However, even these linear experimental laws did not come without some understanding of theoretical concepts of material behavior. Without it one would neither know what experiments to run nor be able to interpret their results. From theoretical viewpoints, the aim of constitutive theories in continuum mechanics is to construct material models consistent with some universal principles so as to enable us, by formulating and analyzing mathematical problems, to predict the outcomes in material behavior verifiable by experimental observations.

We begin with the concept of observer and its role in describing motion and deformation of a material body in the classical Newtonian space-time. An observer, or a frame of reference, is needed to set the stage for mathematical rendering of describing kinematical behaviors of a body, postulating basic laws of mechanics, and formulating constitutive equations of material models.

For the discussion of material behavior and material properties, the concept of objectivity pertaining to the real nature rather than its value affected by different observers is important. This leads to the first universal principle to be discussed, the principle of material frame-indifference, which simply states that the material properties must be independent of observer. This requirement for constitutive equations will be called the condition of material objectivity.

The second universal principle to be discussed deals with the symmetry properties of a material body. Types of materials, such as fluids, solids, and isotropic materials, can be classified according to their symmetry properties.

The conditions of material symmetry and material objectivity are the two major considerations in deducing the restrictions imposed on constitutive equations for a given class of materials. In particular, classes of elastic solids, viscous heat-conducting fluids, and thermoelastic solids are carefully examined, and the classical laws of Hooke, Navier-Stokes, and Fourier are obtained. However, the derivation of these classical laws, already well-known, is not the main purpose of this presentation; rather it provides the mathematical framework for the basic principles in deducing restrictions imposed on general constitutive equations for material models.

Constitutive theories of materials cannot be complete without some thermodynamic considerations. The entropy principle of continuum thermodynamics requires that constitutive equations be consistent with the entropy inequality for any thermodynamic process, and thus like the conditions of material objectivity and material symmetry, it also imposes restrictions on constitutive equations. Since it is not our purpose to present a comprehensive view of thermodynamics in this chapter, we shall illustrate the basic thermodynamic considerations for elastic materials as an example by a relatively simple procedure for exploiting the entropy principle based on the Clausius-Duhem inequality to obtain constitutive restrictions, such as the Gibbs relation, well-known in classical thermostatics, and the Fourier inequality for heat conduction.

The presentations employ the direct notations commonly used in linear algebra, and only in a few occasions the Cartesian components are used. No mathematics beyond linear algebra and differential calculus are needed to understand the physical ideas and most of derivations in the text. Although this is only a chapter in a sequel of many other topics in continuum mechanics, we try to make it as self-contained as possible.

2 Frame of reference, observer

The space-time $\mathcal{W}$ is a four-dimensional space in which physical events occur at some places and certain instants. Let $\mathcal{W}_t$ be the space of placement at the instant t, then the Newtonian space-time of classical mechanics can be expressed as the disjoint union of placement spaces at each instant,

$$\mathcal{W} = \bigcup_{t\in\mathbb{R}} \mathcal{W}_t \,,$$

and therefore, it can be regarded as a product space $\mathbb{E}\times\mathbb{R}$ through a one-to-one mapping

$$\phi : \mathcal{W} \to \mathbb{E}\times\mathbb{R}, \quad \text{such that} \quad \phi_t : \mathcal{W}_t \to \mathbb{E},$$

where $\mathbb{R}$ is the space of real numbers for time and $\mathbb{E}$ is a three-dimensional Euclidean space.

For a Euclidean space $\mathbb{E}$, there is a vector space $\mathbb{V}$, called the translation space of $\mathbb{E}$, such that the difference $\boldsymbol{v} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ of any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ is a vector in $\mathbb{V}$. We also require that the vector space $\mathbb{V}$ be equipped with an inner product, so that length and angle can be defined.

Such a mapping ϕ is called a *frame of reference* and may be regarded as an observer, since it can be depicted as a person taking a snapshot so that the image of ϕ_t is a picture (three-dimensional at least conceptually) of the placements of the events at some instant t, from which the distance between two simultaneous events can be measured. A sequence of events can also be recorded as video clips depicting the change of events in time by an observer.

Now, suppose that two observers are recording the same events with video cameras. In order to compare their video clips regarding the locations and time, they must have a mutual agreement that the clock of their cameras must be synchronized so that simultaneous events can be recognized and since during the recording two observers may move independently while taking pictures with their cameras from different angles, there will be a relative motion and a relative orientation between them. This fact can be expressed mathematically as follows:

Let ϕ and ϕ^* be two observers. We call $* := \phi^* \circ \phi^{-1}$ a *change of frame* (observer),

$$* : \mathbb{E}\times\mathbb{R} \to \mathbb{E}\times\mathbb{R}, \quad *(\boldsymbol{x}, t) = (\boldsymbol{x}^*, t^*),$$

where $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ are the position and time of the same event observed by ϕ and ϕ^* simultaneously. From the observer's agreement, they must be related in the following manner,

$$\boldsymbol{x}^* = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}(t), \quad t^* = t + a, \tag{1}$$

for some constant time difference $a \in \mathbb{R}$, some relative translation $\boldsymbol{c}(t) \in \mathbb{E}$ with respect to the reference point $\boldsymbol{x}_0 \in \mathbb{E}$ and some $Q(t) \in \mathcal{O}(\mathbb{V})$, where $\mathcal{O}(\mathbb{V})$ is the group of orthogonal transformations on the translation space $\mathbb{V}$. In other words, a change of frame is an isometry of space and time as well as preserves the sense of time. Such a transformation will be called a *Euclidean transformation.*

In particular, $\phi_t^* \circ \phi_t^{-1} : \mathbb{E} \to \mathbb{E}$ is given by

$$\boldsymbol{x}^* = \phi_t^*(\phi_t^{-1}(\boldsymbol{x})) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_o) + \boldsymbol{c}(t), \tag{2}$$

which is a time-dependent rigid transformation consisting of an orthogonal transformation and a translation. We shall often call $Q(t)$ the *orthogonal part* of the change of frame from ϕ to ϕ^*.

3 Motion and deformation of a body

In the space-time, the presence of a physical event is represented by its placement at a certain instant so that it can be observed by the observer in a frame of reference. In this sense, the motion of a body can be viewed as a continuous sequence of events such that at any instant t, the placement of the body $\mathcal{B}$ in $\mathcal{W}_t$ is a one-to-one mapping

$$\chi_t : \mathcal{B} \to \mathcal{W}_t.$$

For an observer ϕ with $\phi_t : \mathcal{W}_t \to \mathbb{E}$, the motion can be viewed as a composite mapping $\chi_{\phi_t} := \phi_t \circ \chi_t$,

$$\chi_{\phi_t} : \mathcal{B} \to \mathbb{E}, \quad \boldsymbol{x} = \chi_{\phi_t}(p) = \phi_t(\chi_t(p)), \quad p \in \mathcal{B}.$$

This mapping identifies the body with a region in the Euclidean space, $\mathcal{B}_{\chi_t} := \chi_{\phi_t}(\mathcal{B}) \subset \mathbb{E}$ (see the right part of Figure 1). We call χ_{ϕ_t} a *configuration* of $\mathcal{B}$ at the instant t in the frame ϕ, and a motion of $\mathcal{B}$ is a sequence of configurations of $\mathcal{B}$ in time, $\chi_\phi = \{\chi_{\phi_t},\, t \in \mathbb{R} \mid \chi_{\phi_t} : \mathcal{B} \to \mathbb{E}\}$. We can also express a motion as

$$\chi_\phi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}, \quad \boldsymbol{x} = \chi_\phi(p, t) = \chi_{\phi_t}(p), \quad p \in \mathcal{B}.$$

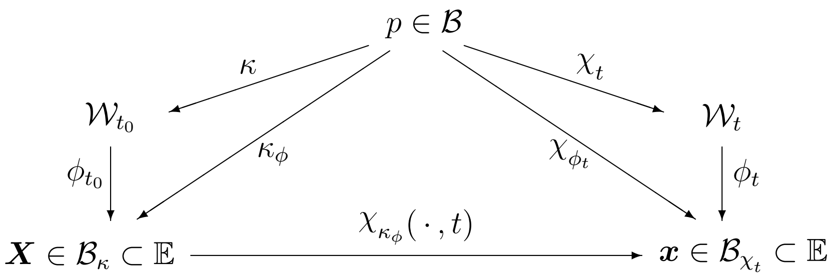

Figure 1: Motion χ_{ϕ_t}, reference configuration κ_ϕ and deformation $\chi_{\kappa_\phi}(\,\cdot\,, t)$

Reference configuration

We regard a body $\mathcal{B}$ as a set of material points. Although it is possible to endow the body as a manifold with a differentiable structure and topology for doing mathematics on the body, to avoid such mathematical subtleties, usually a particular configuration is chosen as reference (see the left part of Figure 1),

$$\kappa_\phi : \mathcal{B} \to \mathbb{E}, \quad \boldsymbol{X} = \kappa_\phi(p), \quad \mathcal{B}_\kappa := \kappa_\phi(\mathcal{B}) \subset \mathbb{E},$$

so that the motion at an instant t is a one-to-one mapping

$$\chi_{\kappa_\phi}(\cdot, t) : \mathcal{B}_\kappa \to \mathcal{B}_{\chi_t}, \quad \boldsymbol{x} = \chi_{\kappa_\phi}(\boldsymbol{X}, t) = \chi_{\phi_t}(\kappa_\phi^{-1}(\boldsymbol{X})), \quad \boldsymbol{X} \in \mathcal{B}_\kappa,$$

defined on a domain in the Euclidean space $\mathbb{E}$ for which topology and differentiability are well defined. This mapping is called a *deformation* from κ to χ_t in the frame ϕ and a motion is then a sequence of deformations in time.

Remember that a configuration is a placement of a body relative to an observer, therefore, for the reference configuration κ_ϕ, there is some instant, say t_0, at which the reference placement κ

of the body is chosen (see Figure 1). On the other hand, the choice of a reference configuration is arbitrary, and it is not necessary that the body should actually occupy the reference place in its motion under consideration. Nevertheless, in most practical problems, t_0 is usually taken as the initial time of the motion.

We can now define kinematic quantities of the motion with domain in $\mathbb{E} \times \mathbb{R}$, such as,

$$\boldsymbol{v}(\boldsymbol{X},t) = \frac{\partial \chi_{\kappa_\phi}(\boldsymbol{X},t)}{\partial t}, \quad \boldsymbol{a}(\boldsymbol{X},t) = \frac{\partial^2 \chi_{\kappa_\phi}(\boldsymbol{X},t)}{\partial t^2}, \quad F(\boldsymbol{X},t) = \nabla_{\boldsymbol{X}} \chi_{\kappa_\phi}(\boldsymbol{X},t).$$

The velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$ are vector quantities, $\boldsymbol{v}, \boldsymbol{a} \in \mathbb{V}$, while the deformation gradient is a non-singular linear transformation, i.e., $F \in \mathcal{L}(\mathbb{V})$, $\det F \neq 0$, since χ_{κ_ϕ} is a one-to-one mapping in $\mathbb{E}$. The space of linear transformations is denoted by $\mathcal{L}(\mathbb{V})$ and a linear transformation is also called a second order tensor or simply a tensor.

Note that no explicit reference to κ_ϕ is indicated in these quantities for brevity, unless similar notations relative to another frame or another reference configuration are involved.

Here of course, we have assumed that $\chi_{\kappa_\phi}(\boldsymbol{X},t)$ is at least twice differentiable with respect to t and once differentiable with respect to $\boldsymbol{X}$. However, from now on, we shall assume that all functions are smooth enough for the conditions needed in the context, without their smoothness explicitly specified.

Material, referential and spatial descriptions

A material body has some physical properties whose values may change along with the deformation of the body in a motion. A quantity defined on a motion can be described in essentially two different ways: either by the evolution of its value along the trajectory of a material point or by the change of its value at a fixed location in space occupied by the body. The former is called a material description and the later a spatial description. We shall make them more precise below. For simplicity, we shall drop the subscript ϕ relative to a fixed frame in the following discussions.

For a given motion $\boldsymbol{x} = \chi(p,t)$, consider a quantity, with its value in some space $\mathbb{W}$, defined on the motion of $\mathcal{B}$ by a function

$$f : \mathcal{B} \times \mathbb{R} \to \mathbb{W}.$$

Then it can also be defined on a reference configuration κ of $\mathcal{B}$,

$$\hat{f} : \mathcal{B}_\kappa \times \mathbb{R} \to \mathbb{W}, \quad \hat{f}(\boldsymbol{X},t) = \hat{f}(\kappa(p),t) = f(p,t). \quad \boldsymbol{X} = \kappa(p), \quad p \in \mathcal{B},$$

and on the position occupied by the body at the present time t,

$$\tilde{f}(\,\cdot\,,t) : \mathcal{B}_t \to \mathbb{W}, \quad \tilde{f}(\boldsymbol{x},t) = \tilde{f}(\chi_\kappa(\boldsymbol{X},t),t) = \hat{f}(\boldsymbol{X},t), \quad \boldsymbol{x} = \chi_\kappa(\boldsymbol{X},t), \quad \boldsymbol{X} \in \mathcal{B}_\kappa.$$

As a custom in continuum mechanics, one usually denotes these functions f, $\hat{f}$ and $\tilde{f}$ by the same symbol since they have the same value at the corresponding point, and write, by an abuse of notations,

$$f = f(p,t) = f(\boldsymbol{X},t) = f(\boldsymbol{x},t),$$

and call them the *material description*, the *referential description* and the *spatial description* of the function f respectively. The referential description is also referred to as the *Lagrangian description* and the spatial description as the *Eulerian description*.

When a reference configuration is chosen and fixed, one can usually identify the material point p with its reference position $\boldsymbol{X}$. In fact, the material description in (p, t) is rarely used and the referential description in $(\boldsymbol{X}, t)$ is often regarded as *the* material description instead. However, in later discussions concerning material frame-indifference of constitutive equations, we shall emphasize the difference between the material description and a referential description, because the true nature of material properties should not depend on the choice of a reference configuration.

Possible confusions may arise due to the abuse of notations when differentiations are involved. To avoid such confusions, we shall use different notations for differentiation in these situations:

$$\dot{f} = \frac{\partial f(\boldsymbol{X}, t)}{\partial t}, \quad \frac{\partial f}{\partial t} = \frac{\partial f(\boldsymbol{x}, t)}{\partial t}, \quad \nabla_{\boldsymbol{X}} f = \frac{\partial f(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}, \quad \nabla_{\boldsymbol{x}} f = \frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}.$$

The relations between these notations can easily be obtained by the chain rule. Indeed, let f be a scalar field and $\boldsymbol{u}$ be a vector field. We have

$$\dot{f} = \frac{\partial f}{\partial t} + (\nabla_{\boldsymbol{x}} f) \cdot \boldsymbol{v}, \quad \dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t} + (\nabla_{\boldsymbol{x}} \boldsymbol{u})\, \boldsymbol{v}, \tag{3}$$

where $\boldsymbol{v}$ is the velocity of the motion, and the referential and spatial gradients are related by

$$\nabla_{\boldsymbol{X}} f = F^T (\nabla_{\boldsymbol{x}} f), \quad \nabla_{\boldsymbol{X}} \boldsymbol{u} = (\nabla_{\boldsymbol{x}} \boldsymbol{u}) F, \tag{4}$$

where the superscript T denotes the transpose of a second order tensor. In spatial description, the gradient and the divergence will also be denoted by the usual notations: $(\operatorname{grad} f)$, $(\operatorname{grad} \boldsymbol{u})$, and $(\operatorname{div} \boldsymbol{u})$.

We call $\dot{f}$ the material time derivative of f, which is the time derivative of f following the trajectory of a material point. Therefore, by the use of (3), the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$ can be expressed as

$$\boldsymbol{v} = \dot{\boldsymbol{x}}, \quad \boldsymbol{a} = \ddot{\boldsymbol{x}} = \dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial t} + (\operatorname{grad} \boldsymbol{v}) \boldsymbol{v}.$$

Moreover, taking the velocity $\boldsymbol{v}$ for $\boldsymbol{u}$ in (4) and since $\nabla_{\boldsymbol{X}} \boldsymbol{v} = \nabla_{\boldsymbol{X}} \dot{\boldsymbol{x}} = \dot{F}$, it follows that

$$L := \operatorname{grad} \boldsymbol{v} = \dot{F} F^{-1}, \tag{5}$$

where L is defined as the spatial gradient of the velocity.

4 Objective tensors

The change of frame (1) on the Euclidean space $\mathbb{E}$ gives rise to a linear mapping on the translation space $\mathbb{V}$, in the following way: Let $\boldsymbol{u}(\phi) = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in \mathbb{V}$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ in the frame ϕ, and $\boldsymbol{u}(\phi^*) = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in \mathbb{V}$ be the corresponding difference vector in the frame ϕ^*, then from (1), it follows immediately that

$$\boldsymbol{u}(\phi^*) = Q(t) \boldsymbol{u}(\phi).$$

Any vector quantity in $\mathbb{V}$, which has this transformation property, is said to be objective with respect to Euclidean transformations, *objective* in the sense that it pertains to a quantity of its real nature rather than its values as affected by different observers. This concept of objectivity can be generalized to any tensor spaces of $\mathbb{V}$.

Definition. *Let* s, $\boldsymbol{u}$, *and* T *be scalar-, vector-, (second order) tensor-valued functions respectively. If relative to a change of frame from* ϕ *to* ϕ^*,

$$s(\phi^*) = s(\phi),$$
$$\boldsymbol{u}(\phi^*) = Q(t)\,\boldsymbol{u}(\phi),$$
$$T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^T,$$

where $Q(t)$ *is the orthogonal part of the change of frame from* ϕ *to* ϕ^*, *then* s, $\boldsymbol{u}$ *and* T *are called* objective *scalar, vector and tensor quantities respectively.*

They are also said to be frame-indifferent with respect to Euclidean transformation or simply Euclidean objective. We call $f(\phi)$ the value of the function observed in the frame ϕ, and for simplicity, often write $f = f(\phi)$ and $f^* = f(\phi^*)$.

The definition of objective scalar is self-evident. For the definition of objective tensors, we consider a scalar $s = \boldsymbol{u} \cdot T\boldsymbol{v}$. Since $(\boldsymbol{u} \cdot T\boldsymbol{v})(\phi) = \boldsymbol{u}(\phi) \cdot T(\phi)\boldsymbol{v}(\phi)$, it follows that

$$\boldsymbol{u}(\phi^*) \cdot T(\phi^*)\boldsymbol{v}(\phi^*) = Q(t)\boldsymbol{u}(\phi) \cdot T(\phi^*)Q(t)\boldsymbol{v}(\phi) = \boldsymbol{u}(\phi) \cdot Q(t)^T T(\phi^*)Q(t)\boldsymbol{v}(\phi),$$

for any objective vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, if $s = \boldsymbol{u} \cdot T\boldsymbol{v}$ is an objective scalar, that is, $\boldsymbol{u}(\phi^*) \cdot T(\phi^*)\boldsymbol{v}(\phi^*) = \boldsymbol{u}(\phi) \cdot T(\phi)\boldsymbol{v}(\phi)$, then it implies that T is an objective tensor satisfying $T(\phi^*) = Q(t)\,T(\phi)\,Q(t)^T$.

One can easily deduce the transformation properties of functions defined on the position and time under a change of frame. Consider an objective scalar field $\psi(\boldsymbol{x}, t) = \psi^*(\boldsymbol{x}^*, t^*)$. Taking the gradient with respect to $\boldsymbol{x}$, from (2) we obtain

$$\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x}, t) = Q(t)^T \nabla_{\boldsymbol{x}^*}\psi^*(\boldsymbol{x}^*, t^*) \quad \text{or} \quad (\operatorname{grad}\psi)(\phi^*) = Q(t)\,(\operatorname{grad}\psi)(\phi),$$

which proves that $(\operatorname{grad}\psi)$ is an objective vector field. Similarly, we can show that if $\boldsymbol{u}$ is an objective vector field then $(\operatorname{grad}\boldsymbol{u})$ is an objective tensor field and $(\operatorname{div}\boldsymbol{u})$ is an objective scalar field. However, one can easily show that the partial derivative $\partial\psi/\partial t$ is not an objective scalar field and neither is $\partial\boldsymbol{u}/\partial t$ an objective vector field.

Transformation properties of motion

Let χ_ϕ be a motion of the body in the frame ϕ, and χ_{ϕ^*} be the corresponding motion in ϕ^*,

$$\boldsymbol{x} = \chi_\phi(p, t), \quad \boldsymbol{x}^* = \chi_{\phi^*}(p, t^*), \quad p \in \mathcal{B}.$$

Then from (2), we have

$$\chi_{\phi^*}(p, t^*) = Q(t)(\chi_\phi(p, t) - \boldsymbol{x}_o) + \boldsymbol{c}(t), \quad p \in \mathcal{B},$$

from which, one can easily show that the velocity and the acceleration are not objective quantities,

$$\dot{\boldsymbol{x}}^* = Q\dot{\boldsymbol{x}} + \dot{Q}(\boldsymbol{x} - \boldsymbol{x}_o) + \dot{\boldsymbol{c}},$$
$$\ddot{\boldsymbol{x}}^* = Q\ddot{\boldsymbol{x}} + 2\dot{Q}\dot{\boldsymbol{x}} + \ddot{Q}(\boldsymbol{x} - \boldsymbol{x}_0) + \ddot{\boldsymbol{c}}. \tag{6}$$

A change of frame (1) with constant $Q(t)$ and $\boldsymbol{c}(t) = \boldsymbol{c}_0 + \boldsymbol{c}_1 t$, for constant $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, is called a *Galilean transformation*. Therefore, from (6) we conclude that the acceleration is not Euclidean objective but it is frame-indifferent with respect to Galilean transformation. Moreover, it also shows that the velocity is neither a Euclidean nor a Galilean objective vector quantity.

Transformation properties of deformation gradient

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Figure 2), then

$$\kappa_\phi = \phi_{t_0} \circ \kappa \quad \text{and} \quad \kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa \tag{7}$$

are the corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant, and

$$\boldsymbol{X} = \kappa_\phi(p), \quad \boldsymbol{X}^* = \kappa_{\phi^*}(p), \quad p \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows from (7) that $\gamma = \phi^*_{t_0} \circ \phi^{-1}_{t_0}$ and by (2), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}(t_0), \tag{8}$$

where $K = Q(t_0)$ is a constant orthogonal tensor.

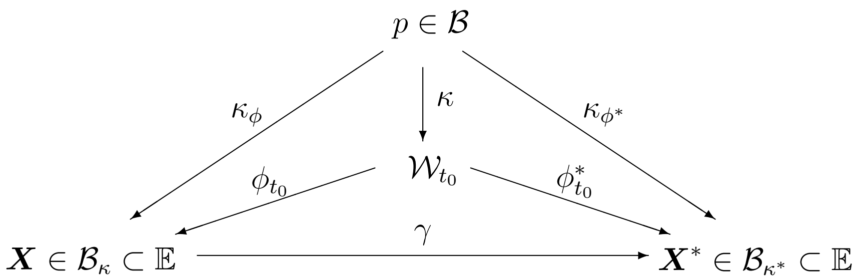

Figure 2: Reference configurations κ_ϕ and κ_{ϕ^*} in the change of frame from ϕ to ϕ^*

On the other hand, the motion in referential description relative to the change of frame is given by $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ and $\boldsymbol{x}^* = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*)$. Hence from (2), we have

$$\chi_{\kappa^*}(\boldsymbol{X}^*, t^*) = Q(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_o) + \boldsymbol{c}(t).$$

Therefore we obtain for the deformation gradient in the frame ϕ^*, i.e., $F^* = \nabla_{\boldsymbol{X}^*}\chi_{\kappa^*}$, by the chain rule and the use of (8),

$$F^*(\boldsymbol{X}^*, t^*) = Q(t)F(\boldsymbol{X}, t)\, Q(t_0)^T, \quad \text{or simply} \quad F^* = QFK^T, \tag{9}$$

where $K = Q(t_0)$ is a constant orthogonal tensor due to the change of frame for the reference configuration.

Remark. The transformation property (9) stands in contrast to $F^* = QF$, the widely used formula which is obtained "provided that the reference configuration is unaffected by the change of frame" as usually implicitly assumed, so that K reduces to the identity transformation. □

The deformation gradient F is not a Euclidean objective tensor. However, the property (9) also shows that it is frame-indifferent with respect to Galilean transformations, since in this case, $K = Q$ is a constant orthogonal transformation.

From (9), we can easily obtain the transformation properties of other kinematic quantities associated with the deformation gradient. In particular, let us consider the velocity gradient defined in (5). We have

$$L^* = \dot{F}^*(F^*)^{-1} = (Q\dot{F} + \dot{Q}F)K^T(QFK^T)^{-1} = (Q\dot{F} + \dot{Q}F)F^{-1}Q^T,$$

which gives

$$L^* = QLQ^T + \dot{Q}Q^T. \tag{10}$$

Moreover, with the decomposition $L = D + W$ into symmetric and skew-symmetric parts, it becomes

$$D^* + W^* = Q(D+W)Q^T + \dot{Q}Q^T.$$

By separating symmetric and skew-symmetric parts, we obtain

$$D^* = QDQ^T, \quad W^* = QWQ^T + \dot{Q}Q^T,$$

since $\dot{Q}Q^T$ is skew-symmetric. Therefore, while the velocity gradient L and the *spin tensor* W are not objective, the *rate of strain tensor* D is an objective tensor.

5 Galilean invariance of balance laws

Motivated by classical mechanics, the balance laws of mass, linear momentum, and energy for deformable bodies,

$$\begin{aligned} &\dot{\rho} + \rho \operatorname{div} \dot{\boldsymbol{x}} = 0, \\ &\rho\, \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\, \boldsymbol{b}, \\ &\rho\, \dot{\varepsilon} + \operatorname{div} \boldsymbol{q} - T \cdot \operatorname{grad} \dot{\boldsymbol{x}} = \rho\, r, \end{aligned} \tag{11}$$

in an inertial frame must be invariant under Galilean transformation. Since by definition, two inertial frames are related by a Galilean transformation, it means that the equations (11) should hold in the same form in any inertial frame. In particular, the balance of linear momentum takes the forms in the inertial frames ϕ and ϕ^*,

$$\rho\, \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\, \boldsymbol{b}, \quad \rho^* \ddot{\boldsymbol{x}}^* - (\operatorname{div} T)^* = \rho^* \boldsymbol{b}^*.$$

Since the acceleration $\ddot{\boldsymbol{x}}$ is Galilean invariant, in order this to hold, it is usually assumed that the mass density ρ, the Cauchy stress tensor T and the body force $\boldsymbol{b}$ are objective scalar, tensor, and vector quantities respectively. Similarly, for the energy equation, it is also assumed that the internal energy ε and the energy supply r are objective scalars, and the heat flux $\boldsymbol{q}$ is an objective vector. These assumptions concern the non-kinematic quantities, including external supplies $(\boldsymbol{b}, r)$, and the constitutive quantities $(T, \boldsymbol{q}, \varepsilon)$.

In fact, for Galilean invariance of the balance laws, only frame-indifference with respect to Galilean transformation for all those non-kinematic quantities would be sufficient. However, traditionally based on physical experiences, it is *postulated* that they are not only Galilean objective but also Euclidean objective. Therefore, with the known transformation properties of the kinematic variables, the balance laws in any arbitrary frame can be deduced.

To emphasize the importance of the objectivity postulate for constitutive theories, it will be referred to as Euclidean objectivity for constitutive quantities:

Euclidean objectivity. *The constitutive quantities: the Cauchy stress T, the heat flux $\boldsymbol{q}$ and the internal energy density ε, are objective (frame-indifferent with respect to Euclidean transformation),*

$$T(\phi^*) = Q(t)\, T(\phi)\, Q(t)^T, \quad \boldsymbol{q}(\phi^*) = Q(t)\, \boldsymbol{q}(\phi), \quad \varepsilon(\phi^*) = \varepsilon(\phi), \tag{12}$$

where $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*.

Note that this postulate concerns only objective properties of balance laws, so that it is a universal property for any material bodies.

6 Constitutive equations in material description

Physically a state of the thermomechanical behavior of a body is characterized by a description of the fields of density $\rho(p,t)$, motion $\chi(p,t)$ and temperature $\theta(p,t)$. The material properties of a body generally depend on the past history of its thermomechanical behavior.

Let us introduce the notion of the past history of a function. Let $h(\cdot)$ be a function of time. The history of h up to time t is defined by

$$h^t(s) = h(t-s),$$

where $s \in [\,0,\infty)$ denotes the time-coordinate pointed into the past from the present time t. Clearly $s = 0$ corresponds to the present time, therefore $h^t(0) = h(t)$.

Mathematical descriptions of material properties are called constitutive equations. We postulate that the history of thermomechanical behavior up to the present time determines the properties of the material body.

Principle of determinism. *Let ϕ be a frame of reference, and $\mathcal{C}$ be a constitutive quantity, then the constitutive equation for $\mathcal{C}$ is given by a functional of the form,*

$$\mathcal{C}(p,t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t, t\,; p), \quad p \in \mathcal{B},\ t \in \mathbb{R}, \tag{13}$$

where the first three arguments are history functions:

$$\rho^t : \mathcal{B} \times [\,0,\infty) \to \mathbb{R}, \quad \chi^t : \mathcal{B} \times [\,0,\infty) \to \mathbb{E}, \quad \theta^t : \mathcal{B} \times [\,0,\infty) \to \mathbb{R}.$$

We call $\mathcal{F}_\phi$ the constitutive function of $\mathcal{C}$ in the frame ϕ. Such a function allows the description of arbitrary non-local effect of an inhomogeneous body with a perfect memory of the past thermomechanical history. With the notation $\mathcal{F}_\phi$, we emphasize that the value of a constitutive function may depend on the observer (frame of reference ϕ).

Constitutive equations can be regarded as mathematical models of material bodies. For a general and rational formulation of constitutive theories, asides from physical experiences, one should rely on some basic principles that a mathematical model should obey lest its consequences be contradictory to physical nature. The most fundamental principles are

- principle of material frame-indifference,
- material symmetry,
- entropy principle.

We shall see in the following sections that these principles impose severe restrictions on material models and hence lead to a great simplification for general constitutive equations. The reduction of constitutive equations to more specific and mathematically simpler ones for a given class of materials is the main objective of constitutive theories in continuum mechanics.

Condition of Euclidean objectivity

Let $\mathcal{C} = \{T, \boldsymbol{q}, \varepsilon\}$ be constitutive quantities and ϕ be a frame of reference. Then from (13), the constitutive equation can be written as

$$\mathcal{C}(p,t) = \mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t, t\,; p), \quad p \in \mathcal{B}, \ \ t \in \mathbb{R}. \tag{14}$$

Similarly, relative to the frame ϕ^*, the corresponding constitutive equation can be written as

$$\mathcal{C}^*(p,t^*) = \mathcal{F}_{\phi^*}((\rho^t)^*, (\mathcal{X}^t)^*, (\theta^t)^*, t^*; p), \quad p \in \mathcal{B}, \ \ t^* \in \mathbb{R}. \tag{15}$$

The two constitutive functions $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi^*}$ are not independent, they must satisfy the Euclidean objectivity relation (12). In particular, for the stress, it implies that

$$\mathcal{F}_{\phi^*}((\rho^t)^*, (\mathcal{X}^t)^*, (\theta^t)^*, t^*; p) = Q(t)\,\mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t, t\,; p)\,Q(t)^T, \quad p \in \mathcal{B}, \tag{16}$$

for any histories $s \in [\,0,\infty)$ and $p' \in \mathcal{B}$, such that

$$\begin{aligned}
&(\rho^t)^*(p',s) = \rho^*(p',t^*-s) = \rho(p',t-s),\\
&(\theta^t)^*(p',s) = \theta^*(p',t^*-s) = \theta(p',t-s),\\
&(\mathcal{X}^t)^*(p',s) = Q(t-s)(\mathcal{X}^t(p',s) - \boldsymbol{x}_o) + \boldsymbol{c}(t-s),
\end{aligned} \tag{17}$$

where $Q(t) \in \mathcal{O}(\mathbb{V})$, $\boldsymbol{x}_o, \boldsymbol{c}(t) \in \mathbb{E}$ are associated with the change of frame from ϕ to ϕ^*.

The first two relations of (17) state that the density and the temperature are objective scalar fields and the last relation follows from the change of frame given by (2).

The relation (16) will be referred to as the *condition of Euclidean objectivity*. It is a relation between the constitutive functions relative to two different frames, and indeed, it determines the constitutive function $\mathcal{F}_{\phi^*}$ once the constitutive function $\mathcal{F}_\phi$ is given.

Example. Consider a material model given by the constitutive equation,

$$T = \mathcal{T}_\phi(\rho, L) = \alpha(\rho)\,I + \beta(\rho)\,L,$$

where I is the identity tensor and L is the velocity gradient. By Euclidean objectivity, $\rho^* = \rho$, $T^* = QTQ^T$, and from the transformation formula (10), $L^* = QLQ^T + \dot{Q}Q^T$, it follows that

$$T^* = Q\,\mathcal{T}_\phi(\rho, L)\,Q^T = \alpha(\rho^*)\,I + \beta(\rho^*)(L^* - \dot{Q}Q^T) := \mathcal{T}_{\phi^*}(\rho^*, L^*),$$

where the constitutive function $\mathcal{T}_{\phi^*}$, defined in the above relation, depends explicitly on $Q(t)$ of the change of frame from ϕ to ϕ^*. From this example, it shows that in general the constitutive functions may depend on the relation between the observers. □

7 Principle of material frame-indifference

It is obvious that *not* any proposed constitutive equations can be used as material models. First of all, they may be frame-dependent in general. However, since the constitutive functions must characterize the intrinsic properties of the material body itself, it should be independent of observer. Consequently, there must be some restrictions imposed on the constitutive functions so that they would be indifferent to the change of frame. This is the essential idea of the principle of material frame-indifference.

Principle of material frame-indifference (in material description)**.** *The constitutive function of an objective (frame-indifferent with respect to Euclidean transformations) constitutive quantity $\mathcal{C}$ must be independent of frame, i.e., for any frames of reference ϕ and ϕ^*, the functionals $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi^*}$, defined by (14) and (15), must have the same form,*

$$\mathcal{F}_\phi(\,\bullet\,;p) = \mathcal{F}_{\phi^*}(\,\bullet\,;p), \quad p \in \mathcal{B}. \tag{18}$$

where $\bullet$ represents the same arguments in both functionals.

This will be referred to as *form-invariance*, $\mathcal{F}_\phi(\,\bullet\,) = \mathcal{F}_{\phi^*}(\,\bullet\,)$, while in (18) the material point p is superfluously indicated to emphasize that it is valid only when the material description is used. The implication of form invariance (18) in referential descriptions will be considered in the next section. Moreover, for a non-objective constitutive quantity, such as the total energy (the kinetic part is not objective), since the quantity itself is frame-dependent, it is obvious that its constitutive function can never be independent of frame.

In our discussions, we shall often consider only the constitutive function of stress, an objective tensor quantity, for simplicity. Similar results can easily be obtained for any other vector or scalar objective constitutive quantities. Thus, from the condition of Euclidean objectivity (16) and the principle of material frame-indifference (18), we obtain the following condition:

Condition of material objectivity. *The constitutive function of an objective tensor constitutive quantity, in material description, satisfies the condition,*

$$\mathcal{F}_\phi((\rho^t)^*, (\mathcal{X}^t)^*, (\theta^t)^*, t^*; p) = Q(t)\, \mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t, t\,; p)\, Q(t)^T, \quad p \in \mathcal{B}, \tag{19}$$

for any histories related by (17) where the change of frame $*$ is arbitrary.

Since the condition (19) involves only the constitutive function in the frame ϕ, it becomes a restriction imposed on the constitutive function $\mathcal{F}_\phi$.

We emphasize that in the condition of Euclidean objectivity (16), $Q(t)$ is the orthogonal part of the change of frame from ϕ to ϕ^*. However, the condition of material objectivity is valid for an arbitrary change of frame from ϕ. Therefore, the condition (19) is valid for any $Q(t) \in \mathcal{O}(\mathbb{V})$.

Sometimes, the condition of material objectivity is referred to as the "*principle* of material objectivity", to impart its relevance in characterizing material property and Euclidean objectivity, as a more explicit form of the principle of material-frame indifference. Indeed, the original principle of material frame-indifference in the fundamental treatise by Truesdell and Noll (1965) was formulated in the form (19).

An immediate restriction imposed by the condition of material objectivity can be obtained by considering a change of frame given by ($Q(t) = 1$, $\boldsymbol{c}(t) = \boldsymbol{x}_0$)

$$\boldsymbol{x}^* = \boldsymbol{x}, \quad t^* = t + a,$$

for arbitrary constant $a \in \mathbb{R}$. By (17), the condition (19) implies that

$$\mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t, t + a\,; p) = \mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t, t\,; p).$$

Since this is true for any value of $a \in \mathbb{R}$, we conclude that $\mathcal{F}_\phi$ can not depend on the argument t and the constitutive equations in general can be expressed as

$$\mathcal{C}(p, t) = \mathcal{F}_\phi(\rho^t, \mathcal{X}^t, \theta^t; p), \quad p \in \mathcal{B}, \ \ t \in \mathbb{R}. \tag{20}$$

Example. In the Example of the previous section, it shows that the constitutive function $T = \mathcal{T}_\phi(\rho, L)$ is frame-dependent. We shall determine the restriction imposed on the constitutive function $\mathcal{T}_\phi$ so that the condition of material objectivity (19) is satisfied. The condition (19) for $\mathcal{T}_\phi$ becomes

$$Q\mathcal{T}_\phi(\rho, L)Q^T = \mathcal{T}_\phi(\rho, QLQ^T + \dot{Q}Q^T) = \mathcal{T}_\phi(\rho, QDQ^T + QWQ^T + \dot{Q}Q^T), \tag{21}$$

which is valid for any orthogonal transformation $Q(t)$, where D and W are the symmetric and skew-symmetric parts of the velocity gradient L. Let us consider a transformation $Q^t(s) = Q(t-s)$ such that it satisfies the following differential equation and the initial condition:

$$\dot{Q}^t(s) + WQ^t(s) = 0, \quad Q^t(0) = I.$$

It is known that the solution exists and can be expressed as $Q^t(s) = \exp(-sW)$ which is an orthogonal transformation because W is skew-symmetric. Clearly it implies that

$$Q(t) = Q^t(0) = I, \quad \dot{Q}(t) = \dot{Q}^t(0) = -W.$$

Therefore, for this choice of orthogonal transformation $Q(t)$ the relation (21) implies that

$$\mathcal{T}_\phi(\rho, L) = \mathcal{T}_\phi(\rho, D).$$

In other words, the condition (19) imposes the restriction that the dependence of the function $\mathcal{T}_\phi$ on L must reduces to the dependence only on the symmetric part of L. □

8 Constitutive equations in referential description

For mathematical analysis, it is more convenient to use referential description so that motions can be defined on the Euclidean space $\mathbb{E}$ instead of the set of material points in $\mathcal{B}$. Therefore, for further discussions, we need to reinterpret the principle of material frame-indifference for constitutive equations relative to a reference configuration.

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Fig. 2), then $\kappa_\phi = \phi_{t_0} \circ \kappa : \mathcal{B} \to \mathbb{E}$ and $\kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa : \mathcal{B} \to \mathbb{E}$ are the two corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant, and

$$\boldsymbol{X} = \kappa_\phi(p) \in \mathbb{E}, \quad \boldsymbol{X}^* = \kappa_{\phi^*}(p) \in \mathbb{E}, \quad p \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then from (8) we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = K(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}(t_0), \tag{22}$$

where $K = \nabla_{\boldsymbol{X}}\gamma = Q(t_0)$ is a constant orthogonal tensor.

The motion in referential description relative to the change of frame is given by

$$\begin{aligned} \boldsymbol{x} = \chi(p,t) \quad &= \chi(\kappa_\phi^{-1}(\boldsymbol{X}), t) \quad = \chi_\kappa(\boldsymbol{X}, t), \qquad \chi = \chi_\kappa \circ \kappa_\phi, \\ \boldsymbol{x}^* = \chi^*(p, t^*) &= \chi^*({\kappa_{\phi^*}}^{-1}(\boldsymbol{X}^*), t^*) = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*), \quad \chi^* = \chi_{\kappa^*} \circ \kappa_{\phi^*}. \end{aligned}$$

From (14) and (15), we can define the corresponding constitutive functions with respect to the reference configuration,

$$\begin{aligned} \mathcal{F}_\phi(\chi^t; p) \quad &= \mathcal{F}_\phi(\chi^t_\kappa \circ \kappa_\phi; p) \quad := \mathcal{H}_\kappa(\chi^t_\kappa; \boldsymbol{X}), \\ \mathcal{F}_{\phi^*}((\chi^t)^*; p) &= \mathcal{F}_{\phi^*}((\chi^t_\kappa)^* \circ \kappa_{\phi^*}; p) := \mathcal{H}_{\kappa^*}((\chi^t_\kappa)^*; \boldsymbol{X}^*), \end{aligned}$$

where only the argument function $\mathcal{X}^t$ is given explicitly while the other variable functions are left out for simplicity. From the above definitions, we can obtain the relation between the constitutive functions $\mathcal{H}_\kappa$ and $\mathcal{H}_{\kappa^*}$ in the referential description,

$$\begin{aligned}\mathcal{H}_{\kappa^*}((\mathcal{X}^t)^*;\boldsymbol{X}^*) &= \mathcal{F}_{\phi^*}((\mathcal{X}^t_\kappa)^*\circ\kappa_{\phi^*};p) = \mathcal{F}_{\phi}((\mathcal{X}^t_\kappa)^*\circ\kappa_{\phi^*};p)\\ &= \mathcal{F}_{\phi}((\mathcal{X}^t_\kappa)^*\circ\gamma\circ\kappa_{\phi};p) = \mathcal{H}_{\kappa}((\mathcal{X}^t)^*\circ\gamma;\boldsymbol{X}),\end{aligned}$$

which begins with the definition and in the second passage the form-invariance (18), $\mathcal{F}_\phi = \mathcal{F}_{\phi^*}$ has been used, and then κ_{ϕ^*} is replaced by $\gamma\circ\kappa_\phi$, and finally the definition again.

Therefore, they are not form-invariant in general, $\mathcal{H}_{\kappa^*}\neq\mathcal{H}_\kappa$, but rather they are related by

$$\mathcal{H}_{\kappa^*}(\,\bullet\,;\boldsymbol{X}^*) = \mathcal{H}_{\kappa}(\,\bullet\circ\gamma\,;\boldsymbol{X}), \tag{23}$$

where $\gamma = \kappa_{\phi^*}\circ\kappa_\phi^{-1}$ is the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame $*$. This is the reinterpretation of the principle of material frame-indifference that in expressing observer independence of material properties, one must also take into account the domains of constitutive functions affected by the change of frame on the reference configuration.

The Euclidean objectivity relation (16) in referential description can be written in the form,

$$\mathcal{H}_{\kappa^*}((\mathcal{X}^t_\kappa)^*;\boldsymbol{X}^*) = Q(t)\,\mathcal{H}_\kappa(\mathcal{X}^t_\kappa;\boldsymbol{X})\,Q(t)^T, \tag{24}$$

where $Q(t)$ is the orthogonal part of the change of frame $*$.

Finally, by combining (23) and (24), we obtain the condition of material objectivity in referential description,

$$\mathcal{H}_{\kappa}((\mathcal{X}^t_\kappa)^*\circ\gamma;\boldsymbol{X}) = Q(t)\,\mathcal{H}_\kappa(\mathcal{X}^t_\kappa;\boldsymbol{X})\,Q(t)^T, \tag{25}$$

valid for any change of frame $*$. In particular, it is valid for any $Q(t)\in\mathcal{O}(\mathbb{V})$.

9 Simple materials

According to the principle of determinism (13), thermomechanical histories of any part of the body can affect the response at any point of the body. In most applications, such a non-local property is irrelevant. Therefore it is usually assumed that only thermomechanical histories in an arbitrary small neighborhood of $\boldsymbol{X}$ affects the material response at the point $\boldsymbol{X}$, and hence the global history functions can be approximated at $\boldsymbol{X}$ by Taylor series up to certain order in a small neighborhood of $\boldsymbol{X}$. In particular, when only linear approximation is concerned, the constitutive equation (20) can be written as

$$\mathcal{C}(\boldsymbol{X},t) = \mathcal{H}_\kappa(\mathcal{X}^t_\kappa(\boldsymbol{X}), F^t(\boldsymbol{X}),\, \theta^t(\boldsymbol{X}),\, \boldsymbol{g}^t(\boldsymbol{X})\,;\boldsymbol{X}), \quad \boldsymbol{X}\in\mathcal{B}_\kappa, \tag{26}$$

where $\boldsymbol{g} = \operatorname{grad}\theta$ is the spatial gradient of temperature. Note that although the constitutive function depends only on local values at the position $\boldsymbol{X}$, it is still general enough to define a material with memory of the body behaviors in the past.

We also remark that since the mass density ρ is related to the determinant of the deformation gradient F and the reference mass density ρ_κ,

$$\rho(\boldsymbol{X},t) = \rho_\kappa(\boldsymbol{X})\;|\det F(\boldsymbol{X},t)|^{-1},$$

the functional dependence of (20) on the history of the mass density ρ is absorbed into the dependence in (26) on $\boldsymbol{X}$ and the history of the deformation gradient F.

An immediate consequence of the condition of material objectivity (25) can be obtained by the following choices of change of frame. Consider a change of frame given by ($Q(t) = 1$, $a = 0$)

$$\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{c}(t) - \boldsymbol{x}_0, \quad t^* = t.$$

Clearly, we have

$$\chi_{\kappa^*}(\gamma(\boldsymbol{X}), t) = \chi_\kappa(\boldsymbol{X}, t) + \boldsymbol{c}(t) - \boldsymbol{x}_0,$$

and the condition (25) implies that

$$\mathcal{H}_\kappa(\chi_\kappa^t + \boldsymbol{c}^t - \boldsymbol{x}_0, F^t; \boldsymbol{X}) = \mathcal{H}_\kappa(\chi_\kappa^t, F^t; \boldsymbol{X}).$$

Since $(\boldsymbol{c}^t(s) - \boldsymbol{x}_0) \in \mathbb{V}$ is arbitrary, we conclude that $\mathcal{H}_\kappa$ can not depend on the history of the trajectory $\chi_\kappa^t(\boldsymbol{X}, s)$ at $\boldsymbol{X}$. In particular, it can not depend on the present position $\boldsymbol{x}$, the velocity $\dot{\boldsymbol{x}}$, and the acceleration $\ddot{\boldsymbol{x}}$.

Therefore, the constitutive equation (26) reduces to

$$\mathcal{C}(\boldsymbol{X}, t) = \mathcal{H}_\kappa(F^t(\boldsymbol{X}), \theta^t(\boldsymbol{X}), \boldsymbol{g}^t(\boldsymbol{X}); \boldsymbol{X}), \quad \boldsymbol{X} \in \mathcal{B}_\kappa, \tag{27}$$

where the thermomechanical variables are history functions up to the present time t at $\boldsymbol{X}$.

A material with constitutive equation (27) is called a *simple material*. The class of simple materials is general enough to include most of the materials of practical interests, such as: elastic solids, thermoelastic solids, viscoelastic solids, as well as elastic fluids, Navier-Stokes fluids and viscous heat-conducting fluids.

For simple materials, by the use of (22), the consequence of form-invariance (23) takes the form,

$$\mathcal{H}_{\kappa^*}((F^t)^*; \boldsymbol{X}^*) = \mathcal{H}_\kappa((F^t)^* K; \boldsymbol{X}),$$

and the Euclidean objectivity condition (24) becomes

$$\mathcal{H}_{\kappa^*}((F^t)^*; \boldsymbol{X}^*) = Q(t)\, \mathcal{H}_\kappa(F^t; \boldsymbol{X})\, Q(t)^T.$$

Combining the above two conditions and knowing the relation, by the use of (9),

$$(F^t)^* K = (Q^t F^t K^T) K = Q^t F^t,$$

we obtain the following condition of material objectivity for simple materials,

$$\mathcal{H}_\kappa(Q^t F^t; \boldsymbol{X}) = Q(t)\, \mathcal{H}_\kappa(F^t; \boldsymbol{X})\, Q(t)^T, \quad \forall\, Q(t) \in \mathcal{O}(\mathbb{V}). \tag{28}$$

Remarks. The condition (28) is the well-known condition of material objectivity, obtained with the assumption that "reference configuration be unaffected by the change of frame" in the fundamental treatise, The Non-Linear Field Theories of Mechanics by Truesdell and Noll (1965). This condition remains valid without such an assumption.

Note that in condition (28), no mention of change of frame is involved, and $Q(t)$ can be interpreted as a superimposed orthogonal transformation on the deformation. This interpretation is sometimes viewed as an alternative version of the principle of material objectivity and is called the "principle of invariance under superimposed rigid body motions", especially, in a purely mechanical constitutive theory. □

10 Material symmetry

We shall consider homogeneous simple material bodies from now on for simplicity. A body is called *homogeneous* in the configuration κ if the constitutive function does not depend on the argument $\boldsymbol{X}$ explicitly,

$$\mathcal{C}(\boldsymbol{X},t) = \mathcal{H}_\kappa(F^t(\boldsymbol{X}),\theta^t(\boldsymbol{X}),\boldsymbol{g}^t(\boldsymbol{X})).$$

In general, constitutive functions may depend on the reference configuration, so let us consider a change of reference configuration. Suppose that $\hat{\kappa}$ is another reference configuration, so that the motion can be written as

$$\boldsymbol{x} = \chi_\kappa(\boldsymbol{X},t) = \chi_{\hat{\kappa}}(\hat{\boldsymbol{X}},t) \quad \text{and} \quad \hat{\boldsymbol{X}} = \xi(\boldsymbol{X}).$$

Let $G = \nabla_{\boldsymbol{X}}\xi \in \mathcal{L}(\mathbb{V})$, then

$$\nabla_{\boldsymbol{X}}\chi_\kappa = (\nabla_{\hat{\boldsymbol{X}}}\chi_{\hat{\kappa}})\,(\nabla_{\boldsymbol{X}}\xi) \quad \text{or} \quad F = \hat{F}G.$$

On the other hand, we have for the temperature and its spatial gradient,

$$\theta(\boldsymbol{X},t) = \hat{\theta}(\hat{\boldsymbol{X}},t), \quad \boldsymbol{g}(\boldsymbol{X},t) = \hat{\boldsymbol{g}}(\hat{\boldsymbol{X}},t).$$

Therefore, from the function $\mathcal{H}_\kappa$, the constitutive function $\mathcal{H}_{\hat{\kappa}}$ relative to the configuration $\hat{\kappa}$ can be defined as

$$\mathcal{H}_\kappa(F^t,\theta^t,\boldsymbol{g}^t) = \mathcal{H}_\kappa(\hat{F}^tG,\hat{\theta}^t,\hat{\boldsymbol{g}}^t) := \mathcal{H}_{\hat{\kappa}}(\hat{F}^t,\hat{\theta}^t,\hat{\boldsymbol{g}}^t). \tag{29}$$

The two functions $\mathcal{H}_\kappa$ and $\mathcal{H}_{\hat{\kappa}}$ are in general different. Consequently, a material body subjected to the same experiment (i.e., the same thermomechanical histories) at two different configurations may have different results.

However, a material body may posses a certain symmetry so that one can not distinguish the outcomes of the same experiments performed at two different configurations. For example, a material body with a cubic crystal structure before and after a rotation of 90° about one of its crystallographic axes is physically indistinguishable.

Definition. *Two reference configurations κ and $\hat{\kappa}$ are said to be materially indistinguishable if their corresponding constitutive functions are the same,*

$$\mathcal{H}_\kappa(\,\bullet\,) = \mathcal{H}_{\hat{\kappa}}(\,\bullet\,).$$

By the second relation of (29), the above condition is equivalent to

$$\mathcal{H}_\kappa(F^t,\theta^t,\boldsymbol{g}^t) = \mathcal{H}_\kappa(F^tG,\theta^t,\boldsymbol{g}^t), \quad \forall\,(F^t,\theta^t,\boldsymbol{g}^t). \tag{30}$$

We call a transformation $G \in \mathcal{L}(\mathbb{V})$ which satisfies (30) a material symmetry transformation with respect to κ.

We assume that a material symmetry transformation is volume-preserving, since, otherwise, if G is a material symmetry transformation, so is G^n for any $n = 1,2,\cdots$, and since $|\det G^n| = |\det G|^n$, the material could suffer arbitrarily large change of volume with no change in material response – a conclusion that seems physically unacceptable. Therefore, we must require that

$G \in \mathcal{U}(\mathbb{V})$, where $\mathcal{U}(\mathbb{V}) = \{G \in \mathcal{L}(\mathbb{V}) : |\det G| = 1\}$ is called the unimodular group on the vector space $\mathbb{V}$.

It is easy to verify that the set of all material symmetry transformations

$$\mathcal{G}_\kappa = \{G \in \mathcal{U}(\mathbb{V}) : \mathcal{H}_\kappa(F^t, \theta^t, \boldsymbol{g}^t) = \mathcal{H}_\kappa(F^t G, \theta^t, \boldsymbol{g}^t), \ \forall\, (F^t, \theta^t, \boldsymbol{g}^t)\}$$

is a subgroup of the unimodular group. We call $\mathcal{G}_\kappa$ the *material symmetry group* of the material body in the reference configuration κ.

Since the relation (30) is a condition concerning only the first argument F^t of the constitutive function, the other arguments will be left out for simplicity in later discussions concerning material symmetry.

Condition of material symmetry. *Constitutive function $\mathcal{H}_\kappa$ must satisfy*

$$\mathcal{H}_\kappa(F^t G) = \mathcal{H}_\kappa(F^t), \quad \forall\, G \in \mathcal{G}_\kappa, \quad \forall\, F^t. \tag{31}$$

Like the condition of material objectivity (28), the condition of material symmetry is also a requirement imposed on the constitutive function $\mathcal{H}_\kappa$.

It is clear that the symmetry group depends on the reference configuration. Suppose that $\hat\kappa$ is another reference configuration such that $\hat{\boldsymbol{X}} = \xi(\boldsymbol{X})$ and $P = \nabla_{\boldsymbol{X}} \xi$. Then for any $G \in \mathcal{G}_\kappa$, from (29) we have

$$\mathcal{H}_{\hat\kappa}(\hat F^t) = \mathcal{H}_\kappa(\hat F^t P) = \mathcal{H}_\kappa(\hat F^t P G) = \mathcal{H}_\kappa(\hat F^t (P G P^{-1}) P) = \mathcal{H}_{\hat\kappa}(\hat F^t (P G P^{-1})),$$

which implies that $PGP^{-1} \in \mathcal{G}_{\hat\kappa}$. Therefore we have proved that

$$\mathcal{G}_{\hat\kappa} = P\, \mathcal{G}_\kappa P^{-1}, \tag{32}$$

where P is the gradient of the change of configuration from κ to $\hat\kappa$, which is a non-singular linear transformation. The groups $\mathcal{G}_\kappa$ and $\mathcal{G}_{\hat\kappa}$ are said to be conjugate to each other.

Physical concepts of real materials such as solids and fluids, can be characterized by their symmetry properties. One of such concepts can be interpreted as saying that a solid has a preferred configuration such that any non-rigid deformation from it alters its material response, while for a fluid any deformation that preserves the density should not affect the material response. The following definitions are based on this concept.

Definition. *A material body is called a simple solid body if the symmetry group is conjugate to an orthogonal subgroup, i.e, if there exists a (preferred) reference configuration κ, such that $\mathcal{G}_\kappa$ is a subgroup of the orthogonal group, $\mathcal{G}_\kappa \subseteq \mathcal{O}(\mathbb{V})$.*

Definition. *A material is called a simple fluid if the symmetry group is the unimodular group, $\mathcal{G}_\kappa = \mathcal{U}(\mathbb{V})$.*

For a simple fluid, the relation (32) implies that the symmetry group is the unimodular group relative to any configuration – a fluid does not have a preferred configuration.

A material which is neither a fluid nor a solid will be called a *fluid crystal*. In other words, for a fluid crystal, the symmetry group is neither the unimodular group nor a group conjugate to any orthogonal subgroup. We shall not consider fluid crystals in this note.

Another concept concerning material symmetry is the material response due to change of orientation.

Definition. *A material body is called isotropic if there exists a configuration κ, such that the symmetry group contains the orthogonal group, $\mathcal{G}_\kappa \supseteq \mathcal{O}(\mathbb{V})$.*

Physically, we can interpret the above definition as saying that any rotation does not alter material response of an isotropic material. The following theorem characterizes isotropic materials (for the proof, see Noll (1965)).

Theorem. *The orthogonal group is maximal in the unimodular group, i.e., if $\mathcal{G}$ is a group such that*

$$\mathcal{O}(\mathbb{V}) \subseteq \mathcal{G} \subseteq \mathcal{U}(\mathbb{V}),$$

then either $\mathcal{G} = \mathcal{O}(\mathbb{V})$ or $\mathcal{G} = \mathcal{U}(\mathbb{V})$.

Therefore, an isotropic material is either a fluid, $\mathcal{G}_\kappa = \mathcal{U}(\mathbb{V})$ for any κ, or an isotropic solid at some configuration κ, $\mathcal{G}_\kappa = \mathcal{O}(\mathbb{V})$. Any other materials are *anisotropic*. Transversely isotropic solids, crystalline solids and fluid crystals are all anisotropic materials. A solid is isotropic at some configuration may not be isotropic at other configurations.

11 Remarks on material models

So far, we have derived the conditions of material objectivity and material symmetry for the stress tensor only for simplicity. For other vector or scalar constitutive quantities, similar results can be easily obtained. We shall summarize the results as follows:

Let $\mathcal{G}$ be the symmetry group of the material and the constitutive equations for the stress T, the heat flux vector $\boldsymbol{q}$, and the internal energy density ε be given by

$$T = \mathcal{T}(F^t, \theta^t, \boldsymbol{g}^t), \quad \boldsymbol{q} = \boldsymbol{q}(F^t, \theta^t, \boldsymbol{g}^t), \quad \varepsilon = \varepsilon(F^t, \theta^t, \boldsymbol{g}^t).$$

- **Condition of material objectivity**

$$\begin{aligned} &\mathcal{T}(Q^tF^t, \theta^t, Q^t\boldsymbol{g}^t) = Q(t)\,\mathcal{T}(F^t, \theta^t, \boldsymbol{g}^t)\,Q(t)^T, \\ &\boldsymbol{q}(Q^tF^t, \theta^t, Q^t\boldsymbol{g}^t) = Q(t)\,\boldsymbol{q}(F^t, \theta^t, \boldsymbol{g}^t), \\ &\varepsilon(Q^tF^t, \theta^t, Q^t\boldsymbol{g}^t) = \varepsilon(F^t, \theta^t, \boldsymbol{g}^t), \end{aligned} \tag{33}$$

  for any $Q^t \in \mathcal{O}(\mathbb{V})$ and any thermomechanical histories $(F^t, \theta^t, \boldsymbol{g}^t)$.

- **Condition of material symmetry**

$$\begin{aligned} &\mathcal{T}(F^tG, \theta^t, \boldsymbol{g}^t) = \mathcal{T}(F^t, \theta^t, \boldsymbol{g}^t), \\ &\boldsymbol{q}(F^tG, \theta^t, \boldsymbol{g}^t) = \boldsymbol{q}(F^t, \theta^t, \boldsymbol{g}^t), \\ &\varepsilon(F^tG, \theta^t, \boldsymbol{g}^t) = \varepsilon(F^t, \theta^t, \boldsymbol{g}^t), \end{aligned} \tag{34}$$

  for any $G \in \mathcal{G}$ and any thermomechanical histories $(F^t, \theta^t, \boldsymbol{g}^t)$.

These conditions are the most fundamental restrictions imposed on any constitutive functions. Since the constitutive functions of simple materials with memory are in general functionals, i.e., functions of functions, the analysis of the above conditions requires much more mathematical hardware and is beyond the context of this chapter.

Therefore, in order to analyze these conditions, we shall restrict ourselves to much simpler material models, namely, simple materials without long range memory. In this case, a history function, say $h^t(s) = h(t-s)$ for small s, can be expressed in the Taylor series approximation,

$$h^t(s) = h(t) - \dot{h}(t)\, s + \frac{1}{2}\, \ddot{h}(t)\, s^2 + \cdots .$$

Therefore, the dependence on the history function can be approximated by the dependence on the values of the function and its derivatives up to a certain order at the present time. With this approximation, constitutive functions become ordinary functions instead of functionals. Constitutive theories of such material models can then be analyzed with linear algebra and differential calculus, no theory of functional analysis will be needed.

We shall consider some material models:

- Elastic materials
  $\mathcal{C} = \mathcal{C}(F)$.
- Thermoelastic materials
  $\mathcal{C} = \mathcal{C}(F, \theta, \boldsymbol{g})$.
- Viscous materials
  $\mathcal{C} = \mathcal{C}(F, \dot{F})$.
- Viscous heat-conducting materials
  $C = \mathcal{C}(F, \dot{F}, \theta, \boldsymbol{g})$.

In the following sections, we shall analyze the restrictions imposed on the constitutive functions, $\mathcal{C} = \{\mathcal{T}, \boldsymbol{q}, \varepsilon\}$, by the conditions of material objectivity and material symmetry on these models for solids and fluids.

Remark on incompressibility

A motion is called *incompressible* if it is volume-preserving, which can be characterized by the condition, $|\det F| = 1$. We call a body an incompressible material body if it is capable of undergoing *only* incompressible motions.

In the discussions of constitutive equations so far, it is assumed that a material body is capable of undergoing any motions. Obviously, for incompressible bodies, some constitutive assumptions must be modified. Indeed, in order to maintain the constant volume in the motion some internal stress is needed to counter the tendency of volume change due to applied forces on the body. This is called the reaction stress which maintains constant volume and hence it should not do any real works in the motion.

Since the rate of work in the motion due to the reaction stress N can be expressed as $(N \cdot \operatorname{grad} \dot{\boldsymbol{x}})$, we shall require that

$$N \cdot \operatorname{grad} \dot{\boldsymbol{x}} = 0.$$

Taking the material time derivative of the equation $\det F = \pm 1$ and by the use of $\operatorname{grad} \dot{\boldsymbol{x}} = \dot{F} F^{-1}$ from (5), we obtain

$$\frac{d}{dt}(\det F) = (\det F) F^{-T} \cdot \dot{F} = (\det F) I \cdot \dot{F} F^{-1} = 0 \quad \text{or} \quad I \cdot \operatorname{grad} \dot{\boldsymbol{x}} = 0.$$

By comparison, we conclude that the reaction stress N must be proportional to the identity tensor, so we can write,

$$N = -p\,I.$$

Therefore for an incompressible body, the stress tensor can be expressed as sum of the reaction pressure and the extra stress,

$$T = -p\,I + \widetilde{\mathcal{T}}(F^t, \theta^t, \boldsymbol{g}^t), \quad |\det F^t| = 1.$$

The *principle of determinism* for an incompressible material body now requires that only the extra stress be given by a constitutive function of thermomechanical histories of the body. Consequently, the constitutive function of the extra stress $\widetilde{\mathcal{T}}$, instead of the (total) stress, is subject to the conditions of material objectivity and material symmetry as discussed in the previous sections.

The reaction pressure p is a function depending on the applied forces on the body and can not be determined entirely by the thermomechanical histories of the body. It is often referred to as *indeterminate* pressure.

12 Elastic materials

Elasticity is a quality of a material body of being able to recover its original state independent of any history of deformation. In other words, the elastic behaviors depend solely on the present state of deformation. Therefore, the mathematical model for the class of elastic materials can be characterized by the constitutive equation for the Cauchy stress tensor as a function of the deformation gradient, $T = \mathcal{T}(F)$.

From (33), the constitutive function must satisfy the conditions of material objectivity,

$$\mathcal{T}(QF) = Q\,\mathcal{T}(F)\,Q^T, \quad \forall\, Q \in \mathcal{O}(\mathbb{V}), \ \forall\, F, \tag{35}$$

and from (34), the condition of material symmetry,

$$\mathcal{T}(FG) = \mathcal{T}(F), \quad \forall\, G \in \mathcal{G}, \ \forall\, F, \tag{36}$$

where $\mathcal{G}$ is the symmetry group of the material body.

Before looking for the restrictions imposed on the constitutive function $\mathcal{T}$, we need a theorem in linear algebra, which plays an important role in the theory of finite deformations. A tensor A is called symmetric positive definite if $A = A^T$ and $\boldsymbol{v} \cdot A\boldsymbol{v} > 0$ for any non-zero vector $\boldsymbol{v} \in \mathbb{V}$.

Theorem (polar decomposition)**.** *For any non-singular tensor F, there exist unique symmetric positive definite tensors U and V and a unique orthogonal tensor R such that*

$$F = R\,U = VR.$$

For the proof, note that since F is non-singular, one can easily show that

$$C = F^T F, \quad B = FF^T,$$

are symmetric positive definite tensors, and hence U and V can be defined uniquely such that

$$C = U^2, \quad B = V^2.$$

It is then easy to verify that $R = FU^{-1}$ is an orthogonal tensor.

We call the symmetric positive definite tensors U and V the right and left *stretch tensors*; C and B the right and left *Cauchy-Green strain tensors* respectively. The orthogonal transformation R is called the *rotation tensor*. These are alternative measures of local deformations.

By the use of polar decomposition $F = RU$, and since the condition of material objectivity (35) is valid for any orthogonal tensor Q, by taking $Q = R^T$, it follows that

$$\mathcal{T}(F) = Q^T \mathcal{T}(QF)\, Q = R\, \mathcal{T}(R^T RU)\, R^T, \quad \text{or} \quad \mathcal{T}(F) = R\, \mathcal{T}(U)\, R^T.$$

Conversely, if

$$\mathcal{T}(F) = R\, \mathcal{T}(U)\, R^T, \quad \forall\, F = RU, \tag{37}$$

let $F^* = QF$ for any $Q \in \mathcal{O}(\mathbb{V})$ and any F, then $F^* = QRU = R^*U^*$, which implies $R^* = QR$ and $U^* = U$ by the uniqueness of polar decomposition. We have from (37) for F^*,

$$\mathcal{T}(QF) = \mathcal{T}(F^*) = QR\, \mathcal{T}(U)\, (QR)^T = Q\, \mathcal{T}(F)\, Q^T.$$

Hence, the condition (35) is satisfied.

Therefore, the constitutive function $\mathcal{T}(F)$ satisfies the condition of material objectivity (35) if and only if it can be represented in the form (37). The representation (37) requires that the dependence on F must reduce to a specific form of dependence on the stretch part U and the rotation part R. It can not depend on the deformation gradient F in an arbitrary manner.

The representation (37) for the Cauchy stress tensor takes a simpler form in terms of the *second Piola-Kirchhoff stress tensor* defined as

$$S = |\det F|\, F^{-1} T F^{-T}.$$

It follows that

$$S = |\det(RU)|\, (RU)^{-1}\, (R\, \mathcal{T}(U)\, R^T)\, (RU)^{-T} = |\det(U)|\, U^{-1} \mathcal{T}(U)\, U^{-T},$$

where $|\det R| = 1$ has been used since $R \in \mathcal{O}(\mathbb{V})$.

Therefore, the material objectivity condition implies that the second Piola-Kirchhoff stress tensor for elastic materials must reduce to a function of the right stretch tensor U or equivalently of the right Cauchy-Green strain tensor $C = U^2$ only,

$$S = \mathcal{S}(C), \quad C = F^T F.$$

This representation is more convenient in practical calculations in terms of the deformation gradient, because no calculation of U from polar decomposition is necessary.

Moreover, from this representation and the condition of material symmetry (36), we have

$$\mathcal{S}(G^T C G) = G^{-1} \mathcal{S}(C)\, G^{-T}, \quad \forall\, G \in \mathcal{G},$$

for any elastic material with symmetry group $\mathcal{G}$.

Note that for any $G \in \mathcal{G}$ the transpose G^T need not belong to $\mathcal{G}$ in general, but if $\mathcal{G} \subseteq \mathcal{O}(\mathbb{V})$, then $G^T = G^{-1}$ must belong to $\mathcal{G}$ by the definition of a group. Therefore, we have the following results for elastic solids.

Elastic solid. *The second Piola-Kirchhoff stress tensor of an elastic solid with symmetry group* $\mathcal{G} \subseteq \mathcal{O}(\mathbb{V})$ *is given by*

$$S = \mathcal{S}(C), \quad \forall\, C = F^T F, \tag{38}$$

for some function $\mathcal{S} : Sym(\mathbb{V}) \to Sym(\mathbb{V})$ *satisfying the following condition:*

$$\mathcal{S}(QCQ^T) = Q\,\mathcal{S}(C)\,Q^T, \quad \forall\, Q \in \mathcal{G},\ \forall\, C. \tag{39}$$

We have denoted the set of all symmetric linear transformation by $Sym(\mathbb{V})$.

A function satisfying the relation (39) is called invariant relative to the group $\mathcal{G}$. Explicit representations for constitutive functions of elastic solids invariant relative to transversely isotropic groups and some symmetry groups of crystalline solids can be found in the literature.

If $\mathcal{G} = \mathcal{O}(\mathbb{V})$ then the relation (39) defines an isotropic invariant function and the body is an isotropic elastic solid. We shall need some representation theorems of isotropic functions for the discussion of isotropic materials in more details later.

12.1 Representations of isotropic functions

Definition. *Let* $\mathcal{D} = \mathbb{R} \times \mathbb{V} \times \mathcal{L}(\mathbb{V})$, *and* $\phi : \mathcal{D} \to \mathbb{R}$, $\boldsymbol{h} : \mathcal{D} \to \mathbb{V}$ *and* $S : \mathcal{D} \to \mathcal{L}(\mathbb{V})$. *We say that* ϕ, $\boldsymbol{h}$, *and* S *are scalar-, vector-, and tensor-valued isotropic functions respectively, if for any* $s \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{V}$, $A \in \mathcal{L}(\mathbb{V})$, *they satisfy the following conditions:*

$$\phi(s, Q\boldsymbol{v}, QAQ^T) = \phi(s, \boldsymbol{v}, A),$$
$$\boldsymbol{h}(s, Q\boldsymbol{v}, QAQ^T) = Q\,\boldsymbol{h}(s, \boldsymbol{v}, A),$$
$$S(s, Q\boldsymbol{v}, QAQ^T) = Q\,S(s, \boldsymbol{v}, A)\,Q^T,$$

for any orthogonal transformation $Q \in \mathcal{O}(\mathbb{V})$.

Isotropic functions are also called isotropic invariants. The definition can easily be extended to any number of scalar, vector and tensor variables. Note that isotropic functions place no restrictions on the scalar variables. Therefore, dependence on scalar variables will be left out in the representation theorems.

Before giving representation theorems for isotropic functions, let us recall a theorem in linear algebra,

Theorem (Cayley-Hamilton). *A linear transformation* $A \in \mathcal{L}(\mathbb{V})$ *satisfies its characteristic equation,*

$$A^3 - I_A A^2 + II_A A - III_A I = 0, \tag{40}$$

where $\{I_A, II_A, III_A\}$ *are called the principal invariants of* A. *They are the coefficients of the characteristic polynomial of* A, *i.e.,*

$$\det(\lambda I - A) = \lambda^3 - I_A \lambda^2 + II_A \lambda - III_A.$$

Since eigenvalues of A are the roots of the characteristic equation, $\det(\lambda I - A) = 0$, if A is symmetric and $\{a_1, a_2, a_3\}$ are three eigenvalues of A, then it follows that

$$I_A = a_1 + a_2 + a_3, \quad II_A = a_1a_2 + a_2a_3 + a_3a_1, \quad III_A = a_1a_2a_3.$$

It is obvious that $I_A = \operatorname{tr} A$ and $III_A = \det A$ are the trace and the determinant of the tensor A respectively. Moreover, I_A, II_A and III_A are respectively a first order, a second order and a third order quantities of $|A|$.

Theorem. *Let $S : Sym(\mathbb{V}) \to Sym(\mathbb{V})$, then it is an isotropic function if and only if it can be represented by*

$$S(A) = s_0 I + s_1 A + s_2 A^2, \tag{41}$$

where s_0, s_1 and s_2 are arbitrary scalar functions of (I_A, II_A, III_A).

Corollary. *If $S(A)$ is an isotropic and linear function of A, then*

$$S(A) = \lambda\,(\mathrm{tr}\,A) I + \mu\, A, \tag{42}$$

where λ and μ are independent of A.

This theorem was first proved by Rivlin & Ericksen (1955). Representations for isotropic functions of any number of vector and tensor variables have been extensively studied and the results are usually tabulated in the literature (see, for example, Wang (1970) and Liu (2002)). We shall give here without proof another theorem for isotropic functions of one vector and one symmetric tensor variables.

Theorem. *Let $\mathcal{D} = \mathbb{V} \times Sym(\mathbb{V})$, and $\phi : \mathcal{D} \to \mathbb{R}$, $\boldsymbol{h} : \mathcal{D} \to \mathbb{V}$, and $S : \mathcal{D} \to Sym(\mathbb{V})$. Then they are isotropic if and only if they can be represented by*

$$\begin{aligned} &\phi = \varphi(I_A, II_A, III_A,\ \boldsymbol{v}\cdot\boldsymbol{v},\ \boldsymbol{v}\cdot A\boldsymbol{v},\ \boldsymbol{v}\cdot A^2\boldsymbol{v}),\\ &\boldsymbol{h} = h_0\boldsymbol{v} + h_1 A\boldsymbol{v} + h_2 A^2\boldsymbol{v},\\ &S = s_0 I + s_1 A + s_2 A^2 + s_3 \boldsymbol{v}\otimes\boldsymbol{v} + s_4(A\boldsymbol{v}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes A\boldsymbol{v}) + s_5 A\boldsymbol{v}\otimes A\boldsymbol{v}, \end{aligned} \tag{43}$$

where the coefficients h_0 through h_2 and s_0 through s_5 are arbitrary functions of the variables indicated in the scalar function φ.

12.2 Hooke's law

In the classical theory of linear elasticity, only small deformations are considered. We introduce the displacement vector from the reference configuration and its gradient,

$$\boldsymbol{u} = \chi_\kappa(\boldsymbol{X}) - \boldsymbol{X}, \quad H = \nabla_{\boldsymbol{X}}\boldsymbol{u}, \quad \boldsymbol{X}\in\mathcal{B}_\kappa.$$

We have $H = F - I$. For small deformations, the displacement gradient H is assumed to be a small quantity of order $o(1)$. The Cauchy-Green tensor C can then be approximated by

$$C = F^T F = (I+H)^T(I+H) = I + H + H^T + H^T H = I + 2E + o(2),$$

where the infinitesimal strain tensor E is defined as the symmetric part of the displacement gradient,

$$E = \frac{1}{2}(H + H^T).$$

The function $\mathcal{S}$ of the equation (38) can now be approximated by

$$\mathcal{S}(C) = \mathcal{S}(I) + \boldsymbol{L}[E] + o(2), \quad \boldsymbol{L}[E] := \frac{d}{dt}\,\mathcal{S}(I + 2E\,t)\Big|_{t=0}.$$

Here we have defined a fourth order tensor $\boldsymbol{L}$ as a linear transformation of $Sym(\mathbb{V})$ into itself. If we further assume that the reference configuration is a natural state, *i.e.*, zero stress at the undeformed state, $\mathcal{T}(I) = 0$, then so is $\mathcal{S}(I) = 0$. Since $\boldsymbol{L}[E]$ is of order $o(1)$ and $F = I + o(1)$, by neglecting the second order terms in (38) we obtain

$$T = \boldsymbol{L}[E], \tag{44}$$

This linear stress-strain relation is known as the *Hooke's law* and $\boldsymbol{L}$ is called the *elasticity tensor*. Since both the stress and the strain tensors are symmetric, by definition, the elasticity tensor has the following symmetry properties in terms of Cartesian components:

$$T_{ij} = \sum_{k,l=1}^{3} L_{ijkl} E_{kl}, \quad L_{ijkl} = L_{jikl} = L_{ijlk}. \tag{45}$$

Moreover, the conditions of material objectivity (35) and material symmetry (36) imply that

$$\mathcal{T}(QFQ^T) = Q\,\mathcal{T}(F)\,Q^T, \quad \forall\, Q \in \mathcal{G} \subseteq \mathcal{O}(\mathbb{V}).$$

Since $\mathcal{T}(F) = \boldsymbol{L}[E(F)]$ and $E(F) = \frac{1}{2}(H + H^T) = \frac{1}{2}(F + F^T) - I$, it follows immediately that

$$\boldsymbol{L}[QEQ^T] = Q\,\boldsymbol{L}[E]\,Q^T, \quad \forall\, Q \in \mathcal{G} \subseteq \mathcal{O}(\mathbb{V}).$$

In other words, the elasticity tensor $\boldsymbol{L} : Sym(\mathbb{V}) \to Sym(\mathbb{V})$ is linear and invariant relative to the symmetry group $\mathcal{G} \subset \mathcal{O}(\mathbb{V})$ for anisotropic linear elastic solids in general. In particular, for $\mathcal{G} = \mathcal{O}(\mathbb{V})$, from the representation (42), the Hooke's law for isotropic linear elastic solid body becomes

$$T = \lambda\,(\operatorname{tr} E) I + \mu\, E,$$

where the material parameters λ and μ are called Lamé elastic coefficients.

12.3 Isotropic elastic solids

For an isotropic elastic solid, the condition of material symmetry (36), $\mathcal{T}(F) = \mathcal{T}(FG)$ is valid for any $G \in \mathcal{G} = \mathcal{O}(\mathbb{V})$. By the use of polar decomposition $F = VR$, and by taking $G = R^T$ which is orthogonal, it follows that

$$\mathcal{T}(F) = \mathcal{T}(V) := \widetilde{\mathcal{T}}(B), \quad B = V^2 = FF^T.$$

It implies that the constitutive function $\mathcal{T}(F)$ must reduce to a function of the left stretch tensor V or the left Cauchy-Green tensor B only. It is independent of the rotation part R of the deformation – an expected result for being isotropic, i.e., the same in all directions.

Moreover, the condition of material objectivity (35) requires the function $\widetilde{\mathcal{T}}(B)$ to satisfy

$$Q\,\widetilde{\mathcal{T}}(B)\,Q^T = \widetilde{\mathcal{T}}((QF)(QF)^T) = \widetilde{\mathcal{T}}(QBQ^T), \quad \forall\, Q \in \mathcal{O}(\mathbb{V}).$$

Conversely, one can easily show that if $\mathcal{T}(F) = \widetilde{\mathcal{T}}(FF^T)$ and if $\widetilde{\mathcal{T}}$ is an isotropic function, then both the conditions of material objectivity and material symmetry are satisfied. Therefore, from the representation theorem (41), the most general constitutive equation for an isotropic elastic solid is given by

$$T = t_0\, I + t_1 B + t_2 B^2,$$

where t_0, t_1, t_2 are functions of the principal invariants $(I_B, I\!I_B, I\!I\!I_B)$.

12.4 Incompressible isotropic elastic solids

Incompressible elastic bodies can be similarly formulated. It has been shown that the reaction stress for incompressibility is an indeterminate pressure and therefore, the constitutive equation for the stress tensor is given by

$$T = -p\,I + t_1 B + t_2 B^2, \quad \det B = III_B = 1,$$

where t_1 and t_2 are functions of (I_B, II_B) and p is the indeterminate pressure. By the use of Cayley-Hamilton theorem, $B^2 = I_B\,B - II_B\,I + B^{-1}$, it can also be expressed by

$$T = -p\,I + s_1 B + s_2 B^{-1}, \quad \det B = 1,$$

where the parameters s_1 and s_2 are functions of (I_B, II_B). Experimental data seem to indicate that

$$s_1 > 0, \quad s_2 \leq 0.$$

Two special cases are of practical interest for finite elasticity, namely, the simple models for which the parameters s_1 and s_2 are constants.

- Neo-Hookean material: $T = -p\,I + s_1 B$.
- Mooney-Rivlin material: $T = -p\,I + s_1 B + s_2 B^{-1}$.

These incompressible material models are often adopted for rubber-like materials. It provides a reasonable theory of natural rubber at finite strains.

13 Viscous fluids

From physical experiences, viscosity is a phenomenon associated with the rate of deformation – the greater the deformation rate, the greater the resistance to motion. Therefore, we shall consider a simple model for viscous materials given by the constitutive equation for the Cauchy stress, $T = \mathcal{T}(F, \dot{F})$.

From (33), the constitutive function $\mathcal{T}$ must satisfy the condition of material objectivity,

$$\mathcal{T}(QF, (QF)^{\cdot}\,) = Q\,\mathcal{T}(F, \dot{F})\,Q^T, \quad \forall\, Q \in \mathcal{O}(\mathbb{V}), \ \forall\, F, \tag{46}$$

and from (34), the condition of material symmetry,

$$\mathcal{T}(FG, \dot{F}G) = \mathcal{T}(F, \dot{F}), \quad \forall\, G \in \mathcal{G}, \ \forall\, F. \tag{47}$$

For fluids, the symmetry group is the unimodular group $\mathcal{G} = \mathcal{U}(\mathbb{V})$. To obtain the restrictions of these conditions imposed on the constitutive function $\mathcal{T}$, we shall take $G = \,|\det F|^{1/3}\,F^{-1}$, obviously $|\det G| = 1$ so that G belongs to the symmetry group, and by (47), it follows that

$$\begin{aligned} \mathcal{T}(F, \dot{F}) &= \mathcal{T}(\,|\det F|^{1/3}\ FF^{-1},\, |\det F|^{1/3}\ \dot{F}F^{-1}) \\ &= \mathcal{T}(\,|\det F|^{1/3}\ I,\, |\det F|^{1/3}\ L) := \widehat{\mathcal{T}}(\,|\det F|\,, L). \end{aligned}$$

Therefore, for fluids, the material symmetry requires that the dependence of $\mathcal{T}$ on $(F, \dot{F})$ be reduced to the dependence on the determinant of F and the velocity gradient $L = \dot{F}F^{-1}$ as defined by the constitutive function $\widehat{\mathcal{T}}(\,|\det F|\,, L)$.

Furthermore, the function $\widehat{\mathcal{T}}$ must satisfy the condition of material objectivity (46) which becomes

$$\widehat{\mathcal{T}}(\,|\det(QF)|\,, (QF)^{\cdot}(QF)^{-1}) = Q\,\widehat{\mathcal{T}}(\,|\det F|\,, L)\,Q^T.$$

Simplifying the left-hand side and decomposing $L = D+W$ into symmetric and skew-symmetric parts, we obtain

$$\widehat{\mathcal{T}}(\,|\det F|\,,\, QDQ^T + QWQ^T + \dot{Q}Q^T) = Q\,\widehat{\mathcal{T}}(\,|\det F|\,, L)\,Q^T. \tag{48}$$

This relation must hold for any orthogonal tensor $Q(t)$. In particular, we can choose it such that $Q(t) = I$ and $\dot{Q}(t) = -W$ (see the Example at the end of Section 7) and the above relation reduces to

$$\widehat{\mathcal{T}}(\,|\det F|\,, L) = \widehat{\mathcal{T}}(\,|\det F|\,, D),$$

where D is the symmetric part of L. Hence, the constitutive function can not depend on the skew-symmetric part W of the velocity gradient. This, in turns, implies from the above relation (48) that $\widehat{\mathcal{T}}(\,|\det F|\,, D)$ is an isotropic tensor function,

$$\widehat{\mathcal{T}}(\,|\det F|\,,\, QDQ^T) = Q\,\widehat{\mathcal{T}}(\,|\det F|\,, D)\,Q^T, \quad \forall\, Q \in \mathcal{O}(\mathbb{V}).$$

Moreover, from the conservation of mass, we have $|\det F| = \rho_\kappa/\rho$, where the mass density ρ_κ in the reference configuration is constant. Consequently, by replacing the dependence on $|\det F|$ with the mass density ρ, and by the use of the representation theorem for isotropic functions (41), we obtain the constitutive equation,

$$T = \widetilde{\mathcal{T}}(\rho, D) = d_0\, I + d_1 D + d_2 D^2, \tag{49}$$

where the material parameters d_0, d_1, d_2 are functions of mass density and three principal invariants of the rate of strain tensor, $(\rho, I_D, I\!I_D, I\!I\!I_D)$.

This is the most general constitutive equation for the viscous fluid of the simple model $\mathcal{T}(F, \dot{F})$. It was first derived by Reiner (1945) and by Rivlin (1947) and it is usually known as *Reiner-Rivlin fluid*. However, we should point out that this is by no means the most general constitutive equation for simple viscous fluids. Indeed, one may consider other viscous fluid models which depend also on deformation rates of higher order, for example, a simple fluid of grade-two $\mathcal{T}(F, \dot{F}, \ddot{F})$. In this note, we shall restrict our attention to simple models only.

13.1 Navier-Stokes fluids

The most well-known viscous fluid models is the Navier-Stokes fluids. It is a mathematically simpler model of the general one (49) in which only linear dependence on the rate of strain is relevant and hence by the linear representation (42), the constitutive equation for Navier-Stokes fluids is given by

$$T = -p(\rho)\, I + \lambda(\rho)(\operatorname{tr} D)\, I + 2\mu(\rho)\, D. \tag{50}$$

The coefficients λ and μ are called the coefficients of viscosity, while μ and $(\lambda + \frac{2}{3}\mu)$ are also known as the *shear* and the *bulk viscosities* respectively. The pressure p and the viscosities λ and μ are functions of ρ.

A Navier-Stokes fluid is also known as a *Newtonian fluid* in fluid mechanics. It is usually assumed that

$$\mu \geq 0, \quad 3\,\lambda + 2\,\mu \geq 0.$$

The non-negativeness of the shear and bulk viscosities can be proved from thermodynamic considerations.

It should be pointed out that unlike the Hooke's law in linear elasticity which is an approximate model for small deformations only, the Navier-Stokes fluids defines a class of material models which satisfies both the conditions of material objectivity and material symmetry. It need not be regarded as the linear approximation of a Reiner-Rivlin fluid. Thus it is conceivable that there are some fluids which obey the constitutive equation (50) for arbitrary rate of deformation. Indeed, water and air are usually treated as Navier-Stokes fluids in most practical applications with very satisfactory results even under rapid flow conditions

13.2 Viscous heat-conducting fluids

We now consider a simple fluid with heat conduction and viscosity given by the following constitutive equation for $\mathcal{C} = \{\mathcal{T}, \boldsymbol{q}, \varepsilon\}$,

$$\mathcal{C} = \mathcal{C}(F, \dot{F}, \theta, \boldsymbol{g}).$$

With the same arguments as before, from the conditions of material objectivity and material symmetry, the constitutive variables $(F, \dot{F}, \theta, \boldsymbol{g})$ must reduce to $(\rho, D, \theta, \boldsymbol{g})$ and the constitutive functions are isotropic functions,

$$\begin{aligned} &\mathcal{T}(\rho, QDQ^T, \theta, Q\,\boldsymbol{g}) = Q\,\mathcal{T}(\rho, D, \theta, \boldsymbol{g})\,Q^T, \\ &\boldsymbol{q}(\rho, QDQ^T, \theta, Q\,\boldsymbol{g}) = Q\,\boldsymbol{q}(\rho, D, \theta, \boldsymbol{g}), \qquad \forall\, Q \in \mathcal{O}(\mathbb{V}), \quad \forall\, (\rho, D, \theta, \boldsymbol{g}). \\ &\varepsilon(\rho, QDQ^T, \theta, Q\,\boldsymbol{g}) = \varepsilon(\rho, D, \theta, \boldsymbol{g}), \end{aligned}$$

Therefore, from the representation (43), one can immediately write down the most general constitutive equations of a viscous heat-conducting fluid for the stress, the heat flux, and the internal energy,

$$\begin{aligned} &T = \alpha_0\, I + \alpha_1 D + \alpha_2\, D^2 + \alpha_3\, \boldsymbol{g} \otimes \boldsymbol{g} + \alpha_4 (D\boldsymbol{g} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes D\boldsymbol{g}) + \alpha_5\, D\boldsymbol{g} \otimes D\boldsymbol{g}, \\ &\boldsymbol{q} = \beta_1\, \boldsymbol{g} + \beta_2\, D\boldsymbol{g} + \beta_3\, D^2\boldsymbol{g}, \\ &\varepsilon = \varepsilon(\rho, \theta, I_D, II_D, III_D,\ \boldsymbol{g}\cdot\boldsymbol{g},\ \boldsymbol{g}\cdot D\boldsymbol{g},\ \boldsymbol{g}\cdot D^2\boldsymbol{g}), \end{aligned}$$

where the coefficient α_i and β_j as well as ε are scalar functions of eight variables indicated in the arguments of ε.

The special case, when only up to linear terms in both the strain rate D and the temperature gradient $\boldsymbol{g}$ are relevant, gives the most widely-used model for viscosity and heat conduction.

Navier-Stokes-Fourier fluids

$$\begin{aligned} &T = -p(\rho, \theta)\, I + \lambda(\rho, \theta)\, (\operatorname{tr} D)\, I + 2\mu(\rho, \theta)\, D, \\ &\boldsymbol{q} = -\kappa(\rho, \theta)\, \boldsymbol{g}, \\ &\varepsilon = \varepsilon(\rho, \theta) + \varepsilon_1(\rho, \theta)\, \operatorname{tr} D. \end{aligned}$$

These are the classical *Navier-Stokes* theory and the *Fourier law* of heat conduction. The material parameters λ, μ are the viscosity coefficients and κ is called the thermal conductivity. From thermodynamic considerations, it is possible to prove that the thermal conductivity is non-negative, and the internal energy is independent of the strain rate, so that $\varepsilon = \varepsilon(\rho, \theta)$.

For incompressible fluids, the mass density ρ is a constant field and the equation of mass balance in (11) implies that $\operatorname{div} \dot{\boldsymbol{x}} = 0$, or $\operatorname{tr} D = 0$. Therefore, we have

Incompressible Navier-Stokes fluids

$$T = -p\,I + 2\mu(\theta)\,D, \quad \operatorname{tr} D = 0,$$
$$\boldsymbol{q} = -\kappa(\theta)\,\boldsymbol{g},$$
$$\varepsilon = \varepsilon(\theta),$$

where the pressure is no longer a constitutive parameter but rather must be determined from the condition $\operatorname{div} \dot{\boldsymbol{x}} = 0$ and suitable boundary conditions.

Another important special case is the simplest model in Continuum Mechanics, given by constitutive equations depending on the density and the temperature only. Hence, it is also a special case of elastic materials.

Elastic fluids

$$T = -p(\rho, \theta)\,I, \quad \boldsymbol{q} = 0, \quad \varepsilon = \varepsilon(\rho, \theta).$$

This defines an inviscid compressible fluid without heat conduction, also known as Euler fluid or ideal fluid in Fluid Mechanics. Similarly, in the case of incompressible Euler fluids, the mass density ρ is a constant field and the indeterminate pressure p depends on boundary conditions and the condition $\operatorname{div} \dot{\boldsymbol{x}} = 0$.

14 Thermodynamic considerations

The constitutive theories of materials cannot be complete without some thermodynamic considerations, which like the conditions of material objectivity and material symmetry, play an important role in imposing restrictions on constitutive equations.

In thermodynamics, two concepts are essential, namely, the energy and the entropy. For the energy, we have already mentioned the equation of energy balance, which is also known as the first law of thermodynamics, while although the entropy seems to be a less intuitive concept, its existence is usually inferred from some more fundamental hypotheses usually known as the second law of thermodynamics. For simplicity, we choose to treat the existence of entropy as a primitive concept and postulate the second law of thermodynamics concerning the evolution of the entropy in the form of an inequality,

$$\rho\dot{\eta} + \operatorname{div} \boldsymbol{\Phi} - \rho s \geq 0. \tag{51}$$

We call η the entropy density, $\boldsymbol{\Phi}$ the entropy flux and s the external entropy supply density. This is called the entropy inequality, which states that the entropy production is a non-negative quantity.

14.1 Entropy principle

One of the principal objectives of continuum mechanics is to determine or predict the behavior of a body once the external supplies are specified. Mathematically, this amounts to solve initial boundary value problems governed by the balance laws of mass, linear momentum and energy,

$$\begin{aligned}
&\dot\rho + \rho \operatorname{div} \dot{\boldsymbol{x}} = 0,\\
&\rho \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho \boldsymbol{b}, \qquad (52)\\
&\rho \dot\varepsilon + \operatorname{div} \boldsymbol{q} - T \cdot \operatorname{grad} \dot{\boldsymbol{x}} = \rho r,
\end{aligned}$$

when the external supplies $\boldsymbol{b}$ and r are given.

The governing field equations are obtained, for the determination of the fields of the density $\rho(\boldsymbol{X}, t)$, the motion $\chi(\boldsymbol{X}, t)$, and the temperature $\theta(\boldsymbol{X}, t)$, after introducing the constitutive equations for T, $\boldsymbol{q}$, and ε, into the balance laws (52), with given external supplies $\boldsymbol{b}(\boldsymbol{X}, t)$ and $r(\boldsymbol{X}, t)$. Any set of fields $\{\rho, \chi, \theta, \boldsymbol{b}, r\}$ satisfying the field equations will be called a *thermodynamic process.*

Mathematically, with constitutive equations of a material model given, one may find a thermodynamic process as a solution satisfying properly prescribed initial and boundary conditions. In this manner of problem solving, a mathematical solution may sometimes lead to physically irrelevant or unrealistic results. This is why we also require thermodynamic processes to be consistent with the second law of thermodynamics to weed out physically unqualified solutions or constitutive models.

Following the idea set forth in the fundamental memoir of Coleman and Noll (1963), the second law of thermodynamics plays the role in the entropy principle as a qualification assessment for constitutive models.

Entropy principle. *It is required that constitutive equations be such that the entropy inequality is satisfied identically for any thermodynamic process.*

From this point of view, like the principle of material objectivity and material symmetry, the entropy principle also imposes restrictions on constitutive functions. To find such restrictions is one of the major tasks in modern continuum thermodynamics.

Motivated by the results of classical thermostatics, it is often assumed that the entropy flux and the entropy supply are proportional to the heat flux and the heat supply respectively. Moreover, both proportional constants are assumed to be the reciprocal of the *absolute temperature* θ,

$$\boldsymbol{\Phi} = \frac{1}{\theta}\,\boldsymbol{q}, \quad s = \frac{1}{\theta}\,r. \qquad (53)$$

The resulting entropy inequality is called the *Clausius-Duhem inequality*,

$$\rho\dot\eta + \operatorname{div}\frac{\boldsymbol{q}}{\theta} - \rho\,\frac{r}{\theta} \geq 0. \qquad (54)$$

Exploitation of entropy principle based on the Clausius-Duhem inequality has been adopted in the development of modern continuum thermodynamics following the Coleman-Noll procedure. We shall illustrate this procedure for thermoelastic material in the following section.

14.2 Thermodynamics of elastic materials

We shall exploit the entropy principle for thermoelastic materials based on the Clausius-Duhem inequality. The constitutive equations for thermoelastic materials can be written as

$$T = T(F, \theta, \boldsymbol{g}), \quad \boldsymbol{q} = \boldsymbol{q}(F, \theta, \boldsymbol{g}), \quad \varepsilon = \varepsilon(F, \theta, \boldsymbol{g}), \quad \eta = \eta(F, \theta, \boldsymbol{g}). \tag{55}$$

Introducing the free energy function,

$$\psi = \varepsilon - \theta\eta,$$

we can rewrite the Clausius-Duhem inequality (54) by eliminating the entropy supply from the energy equation in (52),

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - TF^{-T}\cdot\dot{F} + \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \leq 0,$$

where $\operatorname{grad}\dot{\boldsymbol{x}} = \dot{F}F^{-1}$ has been used. Since $\psi = \psi(F, \theta, \boldsymbol{g})$, the above entropy inequality can be written as

$$H_\theta(F, \theta, \boldsymbol{g})\,\dot{\theta} + H_{\boldsymbol{g}}(F, \theta, \boldsymbol{g})\cdot\dot{\boldsymbol{g}} + H_F(F, \theta, \boldsymbol{g})\cdot\dot{F} + \sigma(F, \theta, \boldsymbol{g}) \leq 0, \tag{56}$$

where

$$H_\theta = \rho\frac{\partial\psi}{\partial\theta} + \rho\eta, \quad H_{\boldsymbol{g}} = \rho\frac{\partial\psi}{\partial\boldsymbol{g}}, \quad H_F = \rho\frac{\partial\psi}{\partial F} - TF^{-T}, \quad \sigma(F, \theta, \boldsymbol{g}) = \frac{1}{\theta}\boldsymbol{q}(F, \theta, \boldsymbol{g})\cdot\boldsymbol{g}.$$

The entropy principle requires that the entropy inequality (56) must hold for any thermodynamic process, i.e., any fields $\{\rho, \chi, \theta, \boldsymbol{b}, r\}$ satisfying the balance laws (52).

Note that for any given fields $\{\chi, \theta\}$, one can determine the fields $\{\rho, \boldsymbol{b}, r\}$ from the balance laws (52), so that $\{\rho, \chi, \theta, \boldsymbol{b}, r\}$ is a thermodynamic process. In particular, for any given values of $\{F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}}\}$ at an arbitrary point $(\boldsymbol{X}_0, t_0)$ such that $\det F \neq 0$ and $\theta > 0$, there exists a thermodynamic process having these values at $(\boldsymbol{X}_0, t_0)$ (see the remarks at Section 14.3).

Consequently, at an arbitrary point, the inequality (56) must hold for arbitrary values of $\{F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}}\}$. In particular, for any given values of $\{F, \theta, \boldsymbol{g}\}$, and $\dot{F} = 0$ and $\dot{\boldsymbol{g}} = 0$, it follows that H_θ and σ are constants and the inequality reduces to

$$H_\theta\,\dot{\theta} + \sigma \leq 0,$$

which must hold for any value of $\dot{\theta}$. Suppose that $H_\theta > 0$, then it will lead to a contradiction, if we choose the value $\dot{\theta} > -\sigma/H_\theta$. For $H_\theta < 0$, choose $\dot{\theta} < -\sigma/H_\theta$ also leads to a contradiction. Hence H_θ must vanish, and by similar arguments, both $H_{\boldsymbol{g}}$ and H_F must vanish too.

Therefore, we have derived the following thermodynamic restrictions on the constitutive functions for thermoelastic materials,

$$T = \rho\frac{\partial\psi}{\partial F}F^T, \quad \varepsilon = \psi - \theta\frac{\partial\psi}{\partial\theta}, \quad \eta = -\frac{\partial\psi}{\partial\theta}. \tag{57}$$

Note that all the constitutive functions in (55) are determined by a single scalar function $\psi = \psi(F, \theta)$, except the constitutive function for the heat flux which satisfies the remaining inequality,

$$\boldsymbol{q}(F, \theta, \boldsymbol{g})\cdot\boldsymbol{g} \leq 0. \tag{58}$$

This is usually referred as the *Fourier inequality* which states that the angle between the heat flux vector and the temperature gradient vector must be more than a right angle, which means that the heat flows in the direction away from the increasing temperature as one would expect.

The existence of the free energy $\psi(F,\theta)$ as the potential function for the stress is usually asserted as saying that the material is *hyperelastic*. The conditions in (57) can be summarized in the following simple relation,

$$d\psi = \frac{1}{\rho} T F^{-T} \cdot dF - \eta \, d\theta,$$

or equivalently, with $\psi = \varepsilon - \theta\eta$, it follows that

$$d\eta = \frac{1}{\theta} \Big(d\varepsilon - \frac{1}{\rho} T F^{-T} \cdot dF \Big). \tag{59}$$

This is the well-known Gibbs relation for elastic materials in classical thermostatics. Here, it is derived in thermodynamics, as a consequence of the entropy principle.

Besides the severe restrictions that the constitutive equations for the stress and the internal energy are completely determined by a single scalar function $\psi(F,\theta)$, the free energy function must also satisfy the conditions of material objectivity and material symmetry,

$$\begin{aligned} &\psi(QF,\theta) = \psi(F,\theta), \quad \forall\, Q \in \mathcal{O}(\mathbb{V}), \\ &\psi(FG,\theta) = \psi(F,\theta), \quad \forall\, G \in \mathcal{G}. \end{aligned} \tag{60}$$

Elastic fluids with heat conduction

For fluids, $\mathcal{G} = \mathcal{U}(\mathbb{V})$, by the same arguments used in Section 13, the condition of material symmetry implies that $\psi(F,\theta)$ depends on the deformation gradient F only through its dependence on $\det F$, or equivalently on the density ρ. Therefore, we obtain from (57)

$$T = -p(\rho,\theta)\, I, \quad \varepsilon = \varepsilon(\rho,\theta),$$

where

$$p = \rho^2 \frac{\partial\psi}{\partial\rho}, \quad \varepsilon = \psi - \theta\, \frac{\partial\psi}{\partial\theta},$$

satisfy the Gibbs relation (59) for elastic fluids,

$$d\eta = \frac{1}{\theta}\Big(d\varepsilon - \frac{p}{\rho^2}\, d\rho \Big),$$

and the integrability condition,

$$\frac{\partial\varepsilon}{\partial\rho} = \frac{p}{\rho^2} - \frac{\theta}{\rho^2} \frac{\partial p}{\partial\theta}.$$

Furthermore, the heat flux $\boldsymbol{q}(F,\theta,\boldsymbol{g})$ must reduce to an isotropic vector function $\boldsymbol{q}(\rho,\theta,\boldsymbol{g})$ for elastic fluid given by

$$\boldsymbol{q} = -\kappa(\rho,\theta)\, \boldsymbol{g}.$$

This is the Fourier law of heat conduction. The Fourier inequality (58) implies that the thermal conductivity κ is non-negative.

Isotropic thermoelastic solids

As proved in Section 12.3, for isotropic elastic solids, $\mathcal{G} = \mathcal{O}(\mathbb{V})$, material symmetry implies that the constitutive dependence on F reduces to the dependence on the Cauchy-Green tensor B, while the condition of material objectivity implies that the constitutive functions are isotropic. Therefore, the constitutive equation for the free energy can be expressed as

$$\psi = \psi(F, \theta) = \widetilde{\psi}(I_B, II_B, III_B, \theta),$$

and the constitutive equations for the stress tensor and the heat flux vector are given by

$$\begin{aligned} T &= t_0\, I + t_1 B + t_2 B^2, \\ \boldsymbol{q} &= \kappa_1\, \boldsymbol{g} + \kappa_2 B\, \boldsymbol{g} + \kappa_3 B^2 \boldsymbol{g}. \end{aligned}$$

The internal energy density ε, the entropy density η, and the coefficients t_0, t_1, t_2, are scalar functions of $(\theta, I_B, II_B, III_B)$, and they can be determined from the potential relations (57),

$$\begin{aligned} t_0 &= 2\rho\, III_B\, \frac{\partial \widetilde{\psi}}{\partial III_B}, \\ t_1 &= 2\rho\Big(\frac{\partial \widetilde{\psi}}{\partial I_B} + I_B \frac{\partial \widetilde{\psi}}{\partial II_B}\Big), \\ t_2 &= -2\rho\, \frac{\partial \widetilde{\psi}}{\partial II_B}. \end{aligned}$$

The coefficients $\kappa_1, \kappa_2, \kappa_3$, in the heat flux are functions of $(\theta, I_B, II_B, III_B,\ \boldsymbol{g}\cdot\boldsymbol{g},\ \boldsymbol{g}\cdot B\boldsymbol{g},\ \boldsymbol{g}\cdot B^2\boldsymbol{g})$ in general.

Linear thermoelasticity

For thermoelastic materials in general, from the condition of material objectivity (60), by decomposing $F = RU$ and choosing $Q = R^T$, we have

$$\psi(F, \theta) = \psi(R^T RU, \theta) = \psi(U, \theta).$$

Therefore, the dependence of the free energy function on the deformation gradient F must reduces to the dependence of the right stretch tensor U.

For the linear theory, the displacement gradient H and the temperature increment $\tilde{\theta} = \theta - \theta_0$ are assumed to be small quantities, where θ_0 is the temperature of the body in the reference state. Since in the linear theory,

$$U = \sqrt{F^T F} = (I + H + H^T + o(2))^{1/2} = I + E + o(2),$$

we can write $\psi = \psi(E, \tilde{\theta})$. Therefore let us express the function ψ up to the second order terms in E and $\tilde{\theta}$ in the following form,

$$\psi = \psi_0 + \psi_1\, \tilde{\theta} + \psi_2\, \tilde{\theta}^2 + \sum_{i,j=1}^{3} M_{ij} E_{ij} + \sum_{i,j=1}^{3} P_{ij} E_{ij}\, \tilde{\theta} + \frac{1}{2\rho_0} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} L_{ijkl} E_{ij} E_{kl}.$$

Moreover, if the reference state is assumed to be a natural state, then M_{ij} must vanish.

The potential relations (57) now takes the form,

$$T = \rho_0 \frac{\partial \psi}{\partial E}, \quad \varepsilon = \psi - (\theta_0 + \tilde{\theta}) \frac{\partial \psi}{\partial \tilde{\theta}},$$

where ρ_0 is the density in the reference state, and we obtain the constitutive equations in the following component forms,

$$T_{ij} = \sum_{k,l=1}^{3} L_{ijkl} E_{kl} + \rho_0 P_{ij}\,\tilde{\theta},$$

$$\varepsilon = \varepsilon_0 + c_v\,\tilde{\theta} - \theta_0 \sum_{i,j=1}^{3} P_{ij} E_{ij},$$

where $\varepsilon_0 = \psi_0 - \theta_0\psi_1$, and $c_v = -2\,\theta_0\psi_2$ is the *specific heat.* Moreover, the coefficients P_{ij} and the elasticity tensor L_{ijkl} have the following symmetry properties,

$$P_{ij} = P_{ji}, \quad L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}.$$

Comparing to the symmetry relations in (45), the additional symmetry between the two pairs of indices (i,j) and (k,l) in the last equality is the consequence of hyperelasticity, i.e., the free energy is the potential function for the stress.

For the linear theory, the heat flux is given by the Fourier law,

$$q_i = -\sum_{j=1}^{3} K_{ij}\, g_j,$$

where K is the thermal conductivity tensor which is positive semi-definite by the Fourier inequality (58), i.e., $\boldsymbol{v}\cdot K\boldsymbol{v} \geq 0$ for any $\boldsymbol{v} \in \mathbb{V}$.

Therefore, we can summarize the constitutive equations of linear thermoelasticity for elastic solids in the following form:

$$T = \boldsymbol{L}[E] + \rho_0(\theta - \theta_0)P,$$
$$\boldsymbol{q} = -K\,\boldsymbol{g},$$
$$\varepsilon = \varepsilon_0 + c_v\,(\theta - \theta_0) - \theta_0\,P\cdot E.$$

If the material is isotropic, then by combining the conditions of material objectivity and material symmetry, for any orthogonal tensor $Q \in \mathcal{G} = \mathcal{O}(\mathbb{V})$, we have

$$T(QFQ^T, \theta) = Q\,T(F,\theta)\,Q^T,$$
$$\boldsymbol{q}(QFQ^T, \theta, Q\boldsymbol{g}) = Q\,\boldsymbol{q}(F,\theta,\boldsymbol{g}).$$

Since $E = \frac{1}{2}(H + H^T) = \frac{1}{2}(F + F^T) - I$, it follows immediately that

$$\boldsymbol{L}[QEQ^T] = Q\,\boldsymbol{L}[E]\,Q^T, \quad P = QPQ^T, \quad KQ = QK,$$

for any orthogonal tensor Q. Since both tensors P and K commute with any orthogonal tensor, they must be constant multiples of the identity tensor, while $\boldsymbol{L}$ is an isotropic linear function of E. Therefore, we conclude that the constitutive equations for isotropic thermoelastic solids are given by

$$T = \lambda\,\mathrm{tr}\,E\,I + 2\mu\,E - \alpha\,(\theta - \theta_0)\,I,$$
$$\boldsymbol{q} = -\kappa\,\boldsymbol{g},$$
$$\varepsilon = \varepsilon_0 + c_v\,(\theta - \theta_0) + \frac{\theta_0}{\rho_0}\,\alpha\ \mathrm{tr}\,E.$$

The elasticity coefficients λ and μ have been defined before in Section 12.2. The thermal conductivity κ is non-negative by the relation (58) . The *thermal expansion* coefficient α and the specific heat c_v are also non-negative quantities, however, it does not follow from the entropy principle, rather from thermal stability analysis, which will not be considered in this chapter.

14.3 Remarks on exploitation of entropy principle

In the previous section, the exploitation of entropy principle based on the Clausius-Duhem inequality gives the most significant restrictions on the constitutive equations of elastic materials consistent with the well-known classical results. This is also true for viscous heat-conducting fluids, in particular, leading to non-negativeness of shear and bulk viscosities. The same exploitation procedure has been adopted in rational thermodynamics for other material classes including materials with memory.

The main assumption (53) on fluxes and supplies based on classical results of thermostatics, while seem to be acceptable in most classical models in continuum thermodynamics, are not particularly well motivated for material models in general. In fact, the assumption (53) is known to be inconsistent with the results from the kinetic theory of ideal gases and is also found to be inappropriate to account for thermodynamics of diffusion and porous media.

In the example presented, the exploitation procedure relies on the assertion that the entropy inequality (56) must hold for any values of $(F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}})$ at an arbitrary point $(\boldsymbol{X}_0, t_0)$. Since the entropy principle requires the entropy inequality to hold for any thermodynamic process, the validity of the above assertion amounts to verify whether there exist a thermodynamic process, i.e., existence of fields $\{\rho, \chi, \theta, \boldsymbol{b}, r\}$ satisfying the balance laws (52) for arbitrarily given values of $(F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}})$ at an arbitrary point $(\boldsymbol{X}_0, t_0)$.

Indeed, in the Coleman-Noll procedure, for arbitrarily given values of $(F_0, \theta_0, \boldsymbol{g}_0, \dot{F}_0, \dot{\theta}_0, \dot{\boldsymbol{g}}_0)$, we can define the fields $\chi(\boldsymbol{X}, t)$ and $\theta(\boldsymbol{X}, t)$ as truncated Taylor series,

$$
\begin{aligned}
&\chi(\boldsymbol{X}, t) = \boldsymbol{X}_0 + F_0(\boldsymbol{X} - \boldsymbol{X}_0) + \dot{F}_0(\boldsymbol{X} - \boldsymbol{X}_0)(t - t_0) + \frac{1}{2}\, \ddot{\boldsymbol{x}}_0 (t - t_0)^2, \\
&\theta(\boldsymbol{X}, t) = \theta_0 + \dot{\theta}_0(t - t_0) + F_0^T \boldsymbol{g}_0 \cdot (\boldsymbol{X} - \boldsymbol{X}_0) + (F_0^T \dot{\boldsymbol{g}}_0 + \dot{F}_0^T \boldsymbol{g}_0) \cdot (\boldsymbol{X} - \boldsymbol{X}_0)(t - t_0).
\end{aligned}
$$

Note that we have used the relations $\boldsymbol{g} = \operatorname{grad}\theta$ and $\nabla_{\boldsymbol{X}}\theta = F^T \boldsymbol{g}$ to set the proper coefficients at the point $(\boldsymbol{X}_0, t_0)$, and the value of the acceleration $\ddot{\boldsymbol{x}}_0$ is determined so as to satisfy the linear momentum balance for arbitrary given value of the external body force $\boldsymbol{b}$ at $(\boldsymbol{X}_0, t_0)$.

From these fields, it is obvious that $(F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}})$ take the given values $(F_0, \theta_0, \boldsymbol{g}_0, \dot{F}_0, \dot{\theta}_0, \dot{\boldsymbol{g}}_0)$ respectively at $(\boldsymbol{X}_0, t_0)$. Furthermore, the mass balance can be integrated to give the density field $\rho(\boldsymbol{X}, t)$, while the balance of energy can be satisfied by a proper choice of the external energy supply r. In this manner we can construct a thermodynamic process consistent with arbitrarily given values of $(F, \theta, \boldsymbol{g}, \dot{F}, \dot{\theta}, \dot{\boldsymbol{g}})$ at $(\boldsymbol{X}_0, t_0)$.

Note that in this construction, although the external body force $\boldsymbol{b}$ can be given arbitrarily, the external energy supply r must be properly chosen so as to satisfy the equation of energy balance. Physically, this argument means that the body must be placed in a proper thermal environment with specific external energy supply field determined by the equation of energy balance. Although this might seem to be acceptable conceptually, it is rather a wishful thinking in the real world.

A better way to account for the above two shortcomings, namely, the specific assumptions (53) and a proper choice of external environment, is to replace the Clausius-Duhem inequality with the general entropy inequality (51). The exploitation of the entropy principle based on the general entropy inequality (51) was proposed by Müller (1971), and the use of Lagrange multipliers by treating the balance laws as constraints on arbitrarily given thermomechanical fields greatly facilitates the exploitation procedure. The procedure is much more elaborate and will not be considered further in this chapter (see for example, Liu (2002)).

Glossary

Frame of reference: A one-to-one mapping which associates an event in space-time to a position in a Euclidean space and a real number in time.

Observer: A person equipped with a ruler and a clock (or any modern equivalent) so that he can measure the distance and the instant of time between simultaneous events.

Objectivity: It pertains to a quantity of its real nature, as a scalar, a vector, or a tensor, rather than its values affected by a change of observer.

Thermomechanical history: Thermal and mechanical behavior, described by temperature and deformation fields of a material body, from past up to the present time.

Constitutive function: Functional description characterizing the response of a material body to thermomechanical history.

Material frame-indifference: Indifference of material response on whoever the observer happens to measure its properties.

Simple material: A class of materials with constitutive functions of local dependence on thermomechanical fields up to the first gradients. The classical materials such as elastic, viscoelastic, Navier-Stokes fluid belong to this class of materials.

Material symmetry: Indifference of material response to the same thermomechanical history relative to certain change of reference state.

Entropy principle: The requirement of constitutive functions of a material body so that its thermomechanical behavior be consistent with the second law of thermodynamics.

Nomenclature

$\mathcal{W}$	The four-dimensional Newtonian space-time.
$\mathcal{W}_t$	The space of placement at the instant t of the Newtonian space-time.
$\mathbb{R}$	The space of real numbers.
$\mathbb{E}$	A three-dimensional Euclidean space.
$\mathbb{V}$	The translation space of the Euclidean space $\mathbb{E}$. It is a vector space with inner product.
$\mathcal{L}(\mathbb{V})$	The space of linear transformation on $\mathbb{V}$. The space of second order tensors.
$\mathcal{O}(\mathbb{V})$	The space of orthogonal transformation on $\mathbb{V}$. It is a group.
$\mathcal{U}(\mathbb{V})$	The group of unimodular linear transformation on $\mathbb{V}$.
$Sym(\mathbb{V})$	The set of symmetric linear transformations in $\mathcal{L}(\mathbb{V})$.
$:=$	Defined as.
$\forall$	For all.
I	Identity tensor in $\mathcal{L}(\mathbb{V})$.
$\boldsymbol{u}\cdot\boldsymbol{v}$	Inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{V}$.
$\boldsymbol{u}\otimes\boldsymbol{v}$	Tensor product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, defined by $(\boldsymbol{u}\otimes\boldsymbol{v})(\boldsymbol{w}) = (\boldsymbol{v}\cdot\boldsymbol{w})\boldsymbol{u}$ for any $\boldsymbol{w}\in\mathbb{V}$.
A^T	Transpose of tensor A.
A^{-T}	Inverse of the transpose of tensor A, i.e., $A^{-T} = (A^T)^{-1} = (A^{-1})^T$.
$\operatorname{tr} A$	Trace of tensor A.
$\det A$	Determinant of tensor A.
$A\cdot B$	Inner product of tensors A and B in $\mathcal{L}(\mathbb{V})$, defined by $A\cdot B = \operatorname{tr}(AB^T)$.
$\mathcal{B}$	A material body.
p	Material point of a body $\mathcal{B}$.
$\boldsymbol{X}$	Material point at the reference configuration of a body $\mathcal{B}$.
$\boldsymbol{x}$	Current position of a material point in $\mathbb{E}$.
$\dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}$	Velocity and acceleration vectors.
χ	Motion of a body, $\boldsymbol{x} = \chi(p,t)$.
F	Deformation gradient, $F = \nabla_{\boldsymbol{X}}\chi$.
B, C	Left and right Cauchy-Green strain tensors, $B = FF^T$ and $C = F^TF$.

V, U	Left and right stretch tensors by polar decomposition $F = VR = RU$.
H	Displacement gradient, $H = F - I$.
E	Infinitesimal strain tensor, $E = (H + H^T)/2$.
L	Velocity gradient, $L = \operatorname{grad} \dot{\boldsymbol{x}}$.
D	Rate of strain tensor, $D = (L + L^T)/2$.
W	Spin tensor, $W = (L - L^T)/2$.
ρ	Mass density.
θ	Temperature.
$\boldsymbol{g}$	Temperature gradient, $\boldsymbol{g} = \operatorname{grad} \theta$.
T	Cauchy stress tensor.
S	Second Piola-Kirchhoff stress tensor, $S = \lvert\det F\rvert\, F^{-1} T F^{-T}$.
$\boldsymbol{q}$	Heat flux vector.
ε	Internal energy density.
ψ	Free energy density.
η	Entropy density.
$\boldsymbol{\Phi}$	Entropy flux vector.
$\boldsymbol{b}$	External body force.
r	External energy supply.
s	External entropy supply.

Bibliography

Coleman, B. D. and Noll, W. (1963). The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Rational Mech. Anal.* **13**, 167-178. [It laid the foundation of modern thermodynamics, based on the idea of entropy inequality as a restriction on constitutive equations.]

Liu, I-Shih (2002). *Continuum Mechanics*, Springer. [It contains a chapter on the entropy principle based on the general entropy inequality and the use of Lagrange multipliers for the exploitation of thermodynamic restrictions on constitutive equations.]

Müller, I. (1971). Die Kältefunktion eine universelle Funktion in der Thermodynamik viskoser wärmeleitender Flüssigkeiten. *Arch. Rational Mech. Anal.* **40**, 1-36. [The general entropy inequality was proposed and analyzed without the assumptions concerning fluxes and supplies between entropy and energy.]

Noll, W. (1965). Proof of the maximality of the orthogonal group in the unimodular group, *Arch. Rational Mech. Anal.* **18**, 100-102. [This result leads to the classification of isotropic and anisotropic material bodies.]

Reiner, M. (1945). A mathematical theory of dilatancy. *Amer. J. Math.* **67**, 350-362. [The first to consider general approach to the derivation of constitutive equations of non-linear viscous fluids.]

Rivlin, R. S. and Ericksen, J. L. (1955). Stress-deformation relations for isotropic materials. *J. Rational Mech. Anal.* **4**, 323-435. [First derivation of the representation for stress as an isotropic tensor function of deformation tensor for nonlinear elastic materials. It had a great influence in the development of finite elasticity.]

Truesdell, C. and Noll, W. (2004). *Non-Linear Field Theories of Mechanics*, Third edition, Edited by S. S. Antman, Springer. [This is the fundamental treatise of continuum mechanics, first published in 1965. It has been the principal sources of references ever since].

Wang, C.-C. (1970). A new representation theorem for isotropic functions, Part I and II. *Arch. Rational Mech. Anal.* **36**, 160-197, 198-223; Corrigendum, ibid, **43**, 392-395 (1971). [The exhaustive derivation of representations for scalar, vector, and tensor isotropic functions. It is a source of reference with tabulated results of isotropic invariant elements.]

Biographical sketch

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