Solutions of the Exercises

Hints and answers of the exercises are listed. Since those who are interested in doing the exercises will most probably begin with the mathematical background materials in linear algebra and tensor analysis, for convenience, the exercises of the appendix are listed first.

For simplicity, we have used the abbreviations, such as $A \overset{(1.30)}{=} B$, meaning $A = B$ by the use of (1.30).

Exercise A.1.1

1) $\beta'^{*} = \{\boldsymbol{e}^1, \boldsymbol{e}^2\} = \{(1, -2), (0, 1)\}$.

2) $[g_{ij}] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad [g^{ij}] = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.

3) Contravariant components: $(v^1, v^2) = (3, -1)$;

   covariant components: $(v_1, v_2) = (1, 1)$.

Exercise A.1.2

1) For any $(x_1, x_2), (y_1, y_2) \in I\!R^2$,

$$(y_1, y_2) \cdot T(x_1, x_2) = (3x_1 + x_2)y_1 + (x_1 + 2x_2)y_2$$
$$= (3y_1 + y_2)x_1 + (y_1 + 2y_2)x_2 = (x_1, x_2) \cdot T(y_1, y_2),$$

   therefore, T is symmetric.

2) $[T_{ij}] = \begin{bmatrix} 3 & 7 \\ 7 & 18 \end{bmatrix}, \quad [T^{ij}] = \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix},$

$$[T_i{}^j] = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}, \quad [T^i{}_j] = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}.$$

   Note that $[T_i{}^j]$ and $[T^i{}_j]$ are not symmetric, and $[T_i{}^j] = [T^i{}_j]^T$.

Exercise A.1.3

We have $\beta^* = \{\boldsymbol{e}^1, \boldsymbol{e}^2\} = \{(1,0),(0,1)\}$, $\bar{\beta}^* = \{\bar{\boldsymbol{e}}^1, \bar{\boldsymbol{e}}^2\} = \{(1,-2),(0,1)\}$, and, by definition, the transformation matrices are given by

$$[M_k{}^j]_{\beta\to\bar{\beta}} = \begin{bmatrix} \bar{\boldsymbol{e}}_1\cdot\boldsymbol{e}^1 & \bar{\boldsymbol{e}}_1\cdot\boldsymbol{e}^2 \\ \bar{\boldsymbol{e}}_2\cdot\boldsymbol{e}^1 & \bar{\boldsymbol{e}}_2\cdot\boldsymbol{e}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$$[M_k{}^j]_{\bar{\beta}^*\to\beta^*} = \begin{bmatrix} \boldsymbol{e}^1\cdot\bar{\boldsymbol{e}}_1 & \boldsymbol{e}^1\cdot\bar{\boldsymbol{e}}_2 \\ \boldsymbol{e}^2\cdot\bar{\boldsymbol{e}}_1 & \boldsymbol{e}^2\cdot\bar{\boldsymbol{e}}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = [M_k{}^j]^T_{\beta\to\bar{\beta}} \,.$$

Exercise A.1.4

We have $M_k{}^j = \bar{\boldsymbol{e}}_k\cdot\boldsymbol{e}^j = A\boldsymbol{e}_k\cdot\boldsymbol{e}^j = \boldsymbol{e}^j\cdot A\boldsymbol{e}_k = A^j{}_k$, by $(A.5)_2$.

Exercise A.1.5

Relative to $\beta' = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \{(1,0),(2,1)\}$, from (A.10)

$T\boldsymbol{e}_1 = (3,1) = \boldsymbol{e}_1 + \boldsymbol{e}_2$ and $T\boldsymbol{e}_2 = (7,4) = -\boldsymbol{e}_1 + 4\boldsymbol{e}_2$.

By definition,

$$(\det T)\,\omega(\boldsymbol{e}_1,\boldsymbol{e}_2) = \omega(T\boldsymbol{e}_1, T\boldsymbol{e}_2)$$
$$= \omega(\boldsymbol{e}_1+\boldsymbol{e}_2, -\boldsymbol{e}_1+4\boldsymbol{e}_2) = \omega(\boldsymbol{e}_1, 4\boldsymbol{e}_2) + \omega(\boldsymbol{e}_2, -\boldsymbol{e}_1) = 5\,\omega(\boldsymbol{e}_1,\boldsymbol{e}_2).$$

Likewise, by definition,

$$(\operatorname{tr} T)\,\omega(\boldsymbol{e}_1,\boldsymbol{e}_2) = \omega(T\boldsymbol{e}_1, \boldsymbol{e}_2) + \omega(\boldsymbol{e}_1, T\boldsymbol{e}_2)$$
$$= \omega(\boldsymbol{e}_1+\boldsymbol{e}_2, \boldsymbol{e}_2) + \omega(\boldsymbol{e}_1, -\boldsymbol{e}_1+4\boldsymbol{e}_2) = 5\,\omega(\boldsymbol{e}_1,\boldsymbol{e}_2).$$

Therefore, $\det T = 5$ and $\operatorname{tr} T = 5$.

Exercise A.1.6

1) $(W\boldsymbol{v})_i = W_{ij}v_j \overset{(A.30)_2}{=} \varepsilon_{ijk}w_k v_j = \varepsilon_{ikj}w_j v_k = -\varepsilon_{ijk}w_j v_k = -(\boldsymbol{w}\times\boldsymbol{v})_i$.

2) $((\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w})_i = \varepsilon_{ijk}(\varepsilon_{jpq}u_p v_q)w_k \overset{(A.25)_1}{=} (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})u_p v_q w_k$
$= u_k w_k v_i - v_k w_k u_i = (\boldsymbol{u}\cdot\boldsymbol{w})v_i - (\boldsymbol{v}\cdot\boldsymbol{w})u_i$.

3) $|\boldsymbol{u}\times\boldsymbol{v}|^2 = (\boldsymbol{u}\times\boldsymbol{v})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\varepsilon_{ijk}u_j v_k)(\varepsilon_{ipq}u_p v_q)$
$\overset{(A.25)_1}{=} (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})u_j u_p v_k v_q$
$= u_j u_j v_k v_k - u_j u_k v_k v_j = |\boldsymbol{u}|^2|\boldsymbol{v}|^2 - |\boldsymbol{u}\cdot\boldsymbol{v}|^2$.

4) The result follows from (3) by the use of $\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}|\,|\boldsymbol{v}|\cos\theta(\boldsymbol{u},\boldsymbol{v})$ and $\sin\theta(\boldsymbol{u},\boldsymbol{v}) \geq 0$.

Exercise A.1.7

1) Verify $(\mathit{1}+A)(\mathit{1}+A)^{-1}=\mathit{1}$ directly.
2) Use item (1) iteratively.

Exercise A.1.8

Verify $(\mathit{1}+\boldsymbol{u}\otimes\boldsymbol{v})(\mathit{1}+\boldsymbol{u}\otimes\boldsymbol{v})^{-1}=\mathit{1}$ directly.

Exercise A.1.9

Since $B=\mathit{1}+A$,

$$\det(B-\lambda\mathit{1})=\det(A-(\lambda-1)\mathit{1})$$
$$\overset{(A.35)}{=}-(\lambda-1)^3+I_A(\lambda-1)^2-II_A(\lambda-1)+III_A$$
$$=-\lambda^3+(3+I_A)\lambda^2-(3+2I_A+II_A)\lambda+(1+I_A+II_A+III_A),$$

and the principal invariants of B follow correspondingly.

The formula for $(\mathit{1}+A)^{-1}$ can easily be obtained from the preceding results and the use of the relation,

$$B^{-1}=(B^2-I_BB+II_B\mathit{1})/a,$$

which follows from Cayley-Hamilton theorem.

Exercise A.1.10

By the spectral theorem,

$$A=\sum_{i=1}^{n}a_i\,\boldsymbol{e}_i\otimes\boldsymbol{e}_i,$$

where a_i, for $i=1,\cdots,n$, are eigenvalues of A, satisfying the characteristic equation,

$$(-\lambda)^n+I_1(-\lambda)^{n-1}+\cdots+I_{n-1}(-\lambda)+I_n=0.$$

To prove the theorem, it is suffice to note that

$$A^k=\sum_{i=1}^{n}(a_i)^k\boldsymbol{e}_i\otimes\boldsymbol{e}_i\quad\forall\,k=1,\cdots,n.$$

Exercise A.1.11

$$C=F^TF=\begin{bmatrix}3&\sqrt{3}&0\\\sqrt{3}&5&0\\0&0&1\end{bmatrix}$$

has the eigenvalues {6,2,1}, with the corresponding eigenvectors,

$$\boldsymbol{e}_1=(1/2,\sqrt{3}/2,0),\quad\boldsymbol{e}_2=(\sqrt{3}/2,-1/2,0),\quad\boldsymbol{e}_3=(0,0,1).$$

Therefore, we can write

$$C = 6\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 2\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3,$$

$$U = \sqrt{6}\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sqrt{2}\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3,$$

and in terms of the standard basis β,

$$U = \sqrt{6}\begin{bmatrix} 1/4 & \sqrt{3}/4 & 0 \\ \sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 3/4 & -\sqrt{3}/4 & 0 \\ -\sqrt{3}/4 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix} 3+\sqrt{3} & 3-\sqrt{3} & 0 \\ 3-\sqrt{3} & 1+3\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}.$$

From $R = FU^{-1}$ and $V = FR^T$, we have

$$R = \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 & 0 \\ 1-\sqrt{3} & 1+\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix},$$

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 & 0 \\ \sqrt{3}-1 & \sqrt{3}+1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Exercise A.2.1

By taking derivative of the identity $QQ^T = \mathit{1}$, it follows that

$$\dot{Q}Q^T + Q\dot{Q}^T = \dot{Q}Q^T + (\dot{Q}Q^T)^T = 0.$$

Exercise A.2.2

For any $\boldsymbol{u} \in V$,

$$\partial_{\boldsymbol{v}}\boldsymbol{h}[\boldsymbol{u}] = (\partial_{\boldsymbol{v}}\boldsymbol{h})\boldsymbol{u} = \frac{d}{dt}\boldsymbol{h}(\boldsymbol{v} + t\boldsymbol{u}, A)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big(\big((\boldsymbol{v} + t\boldsymbol{u}) \cdot A(\boldsymbol{v} + t\boldsymbol{u})\big)A^2(\boldsymbol{v} + t\boldsymbol{u})\Big)\Big|_{t=0}$$

$$= (\boldsymbol{u} \cdot A\boldsymbol{v})A^2\boldsymbol{v} + (\boldsymbol{v} \cdot A\boldsymbol{u})A^2\boldsymbol{v} + (\boldsymbol{v} \cdot A\boldsymbol{v})A^2\boldsymbol{u}$$

$$= (A^2\boldsymbol{v} \otimes A\boldsymbol{v})\boldsymbol{u} + (A^2\boldsymbol{v} \otimes \boldsymbol{v})A\boldsymbol{u} + (\boldsymbol{v} \cdot A\boldsymbol{v})A^2\boldsymbol{u},$$

which implies that

$$\partial_{\boldsymbol{v}}\boldsymbol{h} = A^2\boldsymbol{v} \otimes A\boldsymbol{v} + A^2\boldsymbol{v} \otimes A^T\boldsymbol{v} + (\boldsymbol{v} \cdot A\boldsymbol{v})A^2.$$

Likewise, for any $S \in \mathcal{L}(V)$,

$$\partial_A\boldsymbol{h}[S] = \frac{d}{dt}\boldsymbol{h}(\boldsymbol{v}, A + tS)\Big|_{t=0} = (\boldsymbol{v} \cdot S\boldsymbol{v})A^2\boldsymbol{v} + (\boldsymbol{v} \cdot A\boldsymbol{v})(SA + AS)\boldsymbol{v}.$$

Exercise A.2.3

1) $(\partial_A A^{-1})[S] + o(S) = (A+S)^{-1} - A^{-1} = (\mathit{1} + A^{-1}S)^{-1}A^{-1} - A^{-1}$
$= (\mathit{1} - A^{-1}S + o(S))A^{-1} - A^{-1} = -A^{-1}SA^{-1} + o(S)$,
by the use of Exercise A.1.7.

2) Let $f : A \mapsto \operatorname{tr} A$, then by (A.47), $\partial_A f(A) = \mathit{1}$ and by the chain rule,
$\partial_A \operatorname{tr}(A^{-1})[S] = \partial_A f(A^{-1})[\partial_A A^{-1}[S]] = \mathit{1}[-A^{-1}SA^{-1}] = -(A^{-2})^T[S]$,
from item (1). Recall the notation: $A[S] = A \cdot S = \operatorname{tr} AS^T$.

Exercise A.2.4

1) We can show that
$\partial_A A^k[S] = SA^{k-1} + ASA^{k-2} + A^2SA^{k-3} + \cdots + A^{k-1}S$.
Let $f : A \mapsto \operatorname{tr} A$, then by the chain rule,
$\partial_A \operatorname{tr} A^k[S] = \partial_A f(A^k)[\partial_A A^k[S]]$
$= \operatorname{tr}(SA^{k-1} + ASA^{k-2} + A^2SA^{k-3} + \cdots + A^{k-1}S)$
$= k \operatorname{tr}(A^{k-1})S = k(A^{k-1})^T[S]$.

2) Since, for any λ such that $\det(A + \lambda\mathit{1}) \neq 0$,
$\partial_A \det(A + \lambda\mathit{1}) \overset{(A.48)}{=} \det(A + \lambda\mathit{1})(A + \lambda\mathit{1})^{-T}$
$\overset{(A.35)}{=} \partial_A(\lambda^3 + I_A\lambda^2 + \mathit{II}_A\lambda + \mathit{III}_A)$,
from which we obtain
$\det(A + \lambda\mathit{1})\mathit{1} = (\lambda^2\partial_A I_A + \lambda\,\partial_A \mathit{II}_A + \partial_A \mathit{III}_A)(A + \lambda\mathit{1})^T$
$\overset{(A.35)}{=} (\lambda^3 + I_A\lambda^2 + \mathit{II}_A\lambda + \mathit{III}_A)\mathit{1}$.
The results follow from comparing the coefficients of λ^3, λ^2, and λ.

Exercise A.2.5

Verification in index notations is straightforward.

1) $(S^{ij}u_j)_{,i} = S^{ij}{}_{,i}u_j + S^{ij}u_{j,i}$.
2) $(fS^{ij})_{,j} = S^{ij}f_{,j} + fS^{ij}{}_{,j}$.
3) $(u^iv^j)_{,j} = u^i{}_{,j}v^j + u^iv^j{}_{,j}$.
4) $(u^j{}_{,i})_{,j} = (u^j{}_{,j})_{,i}$.

Exercise A.2.6

The results follow from the condition that the second gradient is symmetric.

1) $e^{ijk}(f_{,k})_{,j} = e^{ijk} f_{,kj} = 0.$

2) $(e^{ijk} v_{k,j})_{,i} = e^{ijk} v_{k,ji} = 0.$

3) It is known that if $\operatorname{curl} \boldsymbol{v} = 0$, then there exists a scalar field ϕ, such that $\boldsymbol{v} = \nabla\phi$. Then,

$$\nabla^2 \boldsymbol{v} = \operatorname{div}(\nabla(\nabla\phi)) = \nabla(\operatorname{div} \nabla\phi) = \nabla(\operatorname{div} \boldsymbol{v}) = 0,$$

by the relation (4) of Exercise A.2.5 and the condition $\operatorname{div} \boldsymbol{v} = 0$.

Exercise A.2.7

Let $\boldsymbol{a}$ be any constant vector field.

1) Since $(\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{a} = (\boldsymbol{v} \otimes \boldsymbol{a})\boldsymbol{n}$,

$$\int_{\partial\mathcal{R}} (\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{a}\, da = \int_{\mathcal{R}} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{a})\, dv = \int_{\mathcal{R}} (\nabla v)\boldsymbol{a}\, dv,$$

by the relation (3) of Exercise A.2.5.

2) $$\int_{\partial\mathcal{R}} (\boldsymbol{v} \otimes S\boldsymbol{n})\boldsymbol{a}\, da = \int_{\partial\mathcal{R}} (\boldsymbol{v} \otimes S^T\boldsymbol{a})\boldsymbol{n}\, da = \int_{\mathcal{R}} \operatorname{div}(\boldsymbol{v} \otimes S^T\boldsymbol{a})\, da.$$

Then the proof follows from the relation,

$$\operatorname{div}(\boldsymbol{v} \otimes S^T\boldsymbol{a}) = (\nabla\boldsymbol{v})S^T\boldsymbol{a} + \boldsymbol{v}(\operatorname{div} S \cdot \boldsymbol{a}) = \big((\nabla\boldsymbol{v})S^T + \boldsymbol{v} \otimes \operatorname{div} S\big)\boldsymbol{a},$$

by (3) of Exercise A.2.5 and (A.75).

Exercise A.2.8

Use the results from Exercise A.2.9 and note that

$$\operatorname{div} \boldsymbol{u} = E_{\langle rr\rangle} + E_{\langle\theta\theta\rangle} + E_{\langle zz\rangle} \quad \text{in cylindrical coordinates,}$$

$$\operatorname{div} \boldsymbol{u} = E_{\langle rr\rangle} + E_{\langle\theta\theta\rangle} + E_{\langle\phi\phi\rangle} \quad \text{in spherical coordinates.}$$

Exercise A.2.9

1) By (A.68) and (A.70),

$$E_{jk} = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x^k} + \frac{\partial u_k}{\partial x^j} - 2\, u_i \mathit{\Gamma}_j{}^i{}_k\Big).$$

In cylindrical coordinate system:

$$E_{rr} = \frac{\partial u_r}{\partial r},$$

$$E_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} + r\, u_r,$$

$$E_{zz} = \frac{\partial u_z}{\partial z},$$

$$E_{r\theta} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{2}{r}\, u_\theta\Big),$$

$$E_{rz} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big),$$

$$E_{\theta z} = \frac{1}{2}\Big(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial \theta}\Big).$$

In spherical coordinate system:

$$E_{rr} = \frac{\partial u_r}{\partial r},$$

$$E_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta} + r\, u_r,$$

$$E_{\phi\phi} = \frac{\partial u_\phi}{\partial \phi} + r \sin^2\theta\, u_r + \sin\theta\, \cos\theta\, u_\theta,$$

$$E_{r\theta} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{2}{r}\, u_\theta\Big),$$

$$E_{r\phi} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{2}{r}\, u_\phi\Big),$$

$$E_{\theta\phi} = \frac{1}{2}\Big(\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - 2\, \cot\theta\, u_\phi\Big).$$

2) In cylindrical coordinate system:

$$E_{\langle rr\rangle} = \frac{\partial u_{\langle r\rangle}}{\partial r},$$

$$E_{\langle \theta\theta\rangle} = \frac{1}{r}\frac{\partial u_{\langle \theta\rangle}}{\partial \theta} + \frac{u_{\langle r\rangle}}{r},$$

$$E_{\langle zz\rangle} = \frac{\partial u_{\langle z\rangle}}{\partial z},$$

$$E_{\langle r\theta\rangle} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial u_{\langle r\rangle}}{\partial \theta} + \frac{\partial u_{\langle \theta\rangle}}{\partial r} - \frac{u_{\langle \theta\rangle}}{r}\Big),$$

$$E_{\langle rz\rangle} = \frac{1}{2}\Big(\frac{\partial u_{\langle r\rangle}}{\partial z} + \frac{\partial u_{\langle z\rangle}}{\partial r}\Big),$$

$$E_{\langle \theta z\rangle} = \frac{1}{2}\Big(\frac{\partial u_{\langle \theta\rangle}}{\partial z} + \frac{1}{r}\frac{\partial u_{\langle z\rangle}}{\partial \theta}\Big).$$

In spherical coordinate system:

$$E_{\langle rr\rangle} = \frac{\partial u_{\langle r\rangle}}{\partial r},$$

$$E_{\langle\theta\theta\rangle} = \frac{1}{r}\frac{\partial u_{\langle\theta\rangle}}{\partial\theta} + \frac{u_{\langle r\rangle}}{r},$$

$$E_{\langle\phi\phi\rangle} = \frac{1}{r\sin\theta}\frac{\partial u_{\langle\phi\rangle}}{\partial\phi} + \frac{u_{\langle r\rangle}}{r} + \cot\theta\,\frac{u_{\langle\theta\rangle}}{r},$$

$$E_{\langle r\theta\rangle} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial u_{\langle r\rangle}}{\partial\theta} + \frac{\partial u_{\langle\theta\rangle}}{\partial r} - \frac{u_{\langle\theta\rangle}}{r}\Big),$$

$$E_{\langle r\phi\rangle} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\frac{\partial u_{\langle r\rangle}}{\partial\phi} + \frac{\partial u_{\langle\phi\rangle}}{\partial r} - \frac{u_{\langle\phi\rangle}}{r}\Big),$$

$$E_{\langle\theta\phi\rangle} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\frac{\partial u_{\langle\theta\rangle}}{\partial\phi} + \frac{1}{r}\frac{\partial u_{\langle\phi\rangle}}{\partial\theta} - \cot\theta\,\frac{u_{\langle\phi\rangle}}{r}\Big).$$

Exercise A.2.10

Similar to (A.71), we have

$$(\operatorname{div} T)^i = T^{ij}{}_{,j} = \frac{\partial T^{ij}}{\partial x^j} + T^{kj}\mathit{\Gamma}_k{}^i{}_j + T^{ik}\mathit{\Gamma}_k{}^j{}_j.$$

1) In cylindrical coordinate system:

$$\begin{aligned}(\operatorname{div} T)^r &= \frac{\partial T^{rr}}{\partial r} + \frac{\partial T^{r\theta}}{\partial\theta} + \frac{\partial T^{rz}}{\partial z} + T^{\theta\theta}\mathit{\Gamma}_\theta{}^r{}_\theta + T^{rr}\mathit{\Gamma}_r{}^\theta{}_\theta \\ &= \frac{\partial T^{rr}}{\partial r} + \frac{\partial T^{r\theta}}{\partial\theta} + \frac{\partial T^{rz}}{\partial z} - r\,T^{\theta\theta} + \frac{1}{r}\,T^{rr},\\ (\operatorname{div} T)^\theta &= \frac{\partial T^{\theta r}}{\partial r} + \frac{\partial T^{\theta\theta}}{\partial\theta} + \frac{\partial T^{\theta z}}{\partial z} + \big(T^{r\theta}\mathit{\Gamma}_r{}^\theta{}_\theta + T^{\theta r}\mathit{\Gamma}_\theta{}^\theta{}_r\big) + T^{\theta r}\mathit{\Gamma}_r{}^\theta{}_\theta \\ &= \frac{\partial T^{\theta r}}{\partial r} + \frac{\partial T^{\theta\theta}}{\partial\theta} + \frac{\partial T^{\theta z}}{\partial z} + \frac{3}{r}\,T^{r\theta},\\ (\operatorname{div} T)^z &= \frac{\partial T^{zr}}{\partial r} + \frac{\partial T^{z\theta}}{\partial\theta} + \frac{\partial T^{zz}}{\partial z} + T^{zr}\mathit{\Gamma}_r{}^\theta{}_\theta \\ &= \frac{\partial T^{zr}}{\partial r} + \frac{\partial T^{z\theta}}{\partial\theta} + \frac{\partial T^{zz}}{\partial z} + \frac{1}{r}\,T^{zr}.\end{aligned}$$

In spherical coordinate system:

$$\begin{aligned}(\operatorname{div} T)^r &= \frac{\partial T^{rr}}{\partial r} + \frac{\partial T^{r\theta}}{\partial\theta} + \frac{\partial T^{r\phi}}{\partial\phi} + \big(T^{\theta\theta}\mathit{\Gamma}_\theta{}^r{}_\theta + T^{\phi\phi}\mathit{\Gamma}_\phi{}^r{}_\phi\big) \\ &\quad + \big(T^{rr}\mathit{\Gamma}_r{}^\theta{}_\theta + T^{rr}\mathit{\Gamma}_r{}^\phi{}_\phi + T^{r\theta}\mathit{\Gamma}_\theta{}^\phi{}_\phi\big) \\ &= \frac{\partial T^{rr}}{\partial r} + \frac{\partial T^{r\theta}}{\partial\theta} + \frac{\partial T^{r\phi}}{\partial\phi} + \frac{2}{r}\,T^{rr} - r\,T^{\theta\theta} - r\sin^2\theta\,T^{\phi\phi} + \cot\theta\,T^{r\theta},\end{aligned}$$

$$
\begin{aligned}
(\operatorname{div} T)^\theta &= \frac{\partial T^{\theta r}}{\partial r} + \frac{\partial T^{\theta\theta}}{\partial \theta} + \frac{\partial T^{\theta\phi}}{\partial \phi} + \left(T^{r\theta}\Gamma_{r\ \theta}^{\ \theta} + T^{\theta r}\Gamma_{\theta\ r}^{\ \theta} + T^{\phi\phi}\Gamma_{\phi\ \phi}^{\ \theta}\right) \\
&\quad + \left(T^{\theta r}\Gamma_{r\ \theta}^{\ \theta} + T^{\theta r}\Gamma_{r\ \phi}^{\ \phi} + T^{\theta\theta}\Gamma_{\theta\ \phi}^{\ \phi}\right) \\
&= \frac{\partial T^{\theta r}}{\partial r} + \frac{\partial T^{\theta\theta}}{\partial \theta} + \frac{\partial T^{\theta\phi}}{\partial \phi} + \frac{4}{r}\,T^{\theta r} + \cot\theta\, T^{\theta\theta} - \sin\theta\cos\theta\, T^{\phi\phi}, \\
(\operatorname{div} T)^\phi &= \frac{\partial T^{\phi r}}{\partial r} + \frac{\partial T^{\phi\theta}}{\partial \theta} + \frac{\partial T^{\phi\phi}}{\partial \phi} + \left(T^{r\phi}\Gamma_{r\ \phi}^{\ \phi} + T^{\phi r}\Gamma_{\phi\ r}^{\ \phi} + T^{\theta\phi}\Gamma_{\theta\ \phi}^{\ \phi}\right. \\
&\quad \left. + T^{\phi\theta}\Gamma_{\phi\ \theta}^{\ \phi}\right) + \left(T^{\phi r}\Gamma_{r\ \theta}^{\ \theta} + T^{\phi r}\Gamma_{r\ \phi}^{\ \phi} + T^{\phi\theta}\Gamma_{\theta\ \phi}^{\ \phi}\right) \\
&= \frac{\partial T^{\phi r}}{\partial r} + \frac{\partial T^{\phi\theta}}{\partial \theta} + \frac{\partial T^{\phi\phi}}{\partial \phi} + \frac{4}{r}\,T^{r\phi} + 3\cot\theta\, T^{\theta\phi}.
\end{aligned}
$$

Exercise A.2.11

Let $\Phi : t \mapsto x$, then $x^i = \phi(t)$ and by (A.56),

$x = \Phi(t) = \chi(\phi^1(t), \cdots, \phi^n(t))$.

1) $\dot{\Phi}(t) = \dfrac{\partial \chi}{\partial x^i}\Big|_x \dot{\phi}^i(t) \overset{(A.60)}{=} \dot{\phi}^i(t)\, \boldsymbol{e}_i(\Phi(t))$.

2) From (1),

$$
\begin{aligned}
\ddot{\Phi}(t) &= \ddot{\phi}^j(t)\, \boldsymbol{e}_j(x) + \dot{\phi}^j(t) \nabla \boldsymbol{e}_j(x)\, \dot{\Phi}(t) \\
&\overset{(A.65)}{=} \ddot{\phi}^i(t) \boldsymbol{e}_i(x) + \dot{\phi}^j(t)\ (\Gamma_{j\ l}^{\ i}(x)\, \boldsymbol{e}_i(x) \otimes \boldsymbol{e}^l(x))\ (\dot{\phi}^k(t)\, \boldsymbol{e}_k(x)) \\
&= \left(\ddot{\phi}^i(t) + \dot{\phi}^j(t)\dot{\phi}^k(t)\Gamma_{j\ k}^{\ i}(\Phi(t)\right) \boldsymbol{e}_i(\Phi(t)).
\end{aligned}
$$

Exercise 1.2.1

For coordinate systems (r, θ, z) and (R, Θ, Z), the metric tensors are given by

$$
[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [g^{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

By (1.6), we have

$$
[F^i_{\ \alpha}] = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & 1/a \end{bmatrix}, \text{ and in physical components,}
$$

$$
[F_{\langle i\alpha\rangle}] = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & r/R & \tau r \\ 0 & 0 & 1/a \end{bmatrix} = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{a} & \tau r \\ 0 & 0 & 1/a \end{bmatrix}.
$$

Since $\det F = \det[F_{\langle i\alpha\rangle}] = 1$, the deformation is volume preserving – note that $\det[F^i{}_\alpha] = 1/\sqrt{a} \neq 1$ in general. For $a = 1$, it is a simple shear of amount τr locally in the coordinate surface θ-z at every fixed radius r (compared with (1.15)).

Moreover, we have

$$[B^{ij}] = [F^i{}_\alpha][g^{\alpha\beta}][F^j{}_\beta]^T = \begin{bmatrix} a & 0 & 0 \\ 0 & \tau^2 + a/r^2 & \tau/a \\ 0 & \tau/a & 1/a^2 \end{bmatrix},$$

$$[C_{\alpha\beta}] = [F^i{}_\alpha]^T[g_{ij}][F^j{}_\beta] = \begin{bmatrix} a & 0 & 0 \\ 0 & r^2 & \tau r^2 \\ 0 & \tau r^2 & \tau^2 r^2 + 1/a^2 \end{bmatrix},$$

and in physical components,

$$[B_{\langle ij\rangle}] = \begin{bmatrix} a & 0 & 0 \\ 0 & \tau^2 r^2 + a & \tau r/a \\ 0 & \tau r/a & 1/a^2 \end{bmatrix}, \quad [C_{\langle\alpha\beta\rangle}] = \begin{bmatrix} a & 0 & 0 \\ 0 & a & \sqrt{a}\,\tau r \\ 0 & \sqrt{a}\,\tau r & \tau^2 r^2 + 1/a^2 \end{bmatrix}.$$

One can easily check that $\det[B_{\langle ij\rangle}] = \det[C_{\langle\alpha\beta\rangle}] = 1$.

Remark: It would be of interest to calculate also the matrix $[F^\alpha_\beta]$ disregarding F as a two-point tensor.

To begin with, from $F = F^i{}_\alpha \boldsymbol{e}_i(\boldsymbol{x}) \otimes \boldsymbol{e}^\alpha(\boldsymbol{X})$, we have $F\boldsymbol{e}_\beta = F^i{}_\beta \boldsymbol{e}_i$ and hence, $F^\alpha_\beta = \boldsymbol{e}^\alpha \cdot F\boldsymbol{e}_\beta = (\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_i)F^i{}_\beta$, or $[F^\alpha_\beta] = [\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_i][F^i{}_\beta]$.

One the other hand, one can show that

$$\boldsymbol{e}_r = \cos(\tau Z)\boldsymbol{e}_R + \tfrac{1}{R}\sin(\tau Z)\boldsymbol{e}_\Theta,$$

$$\boldsymbol{e}_\theta = -r\sin(\tau Z)\boldsymbol{e}_R + \tfrac{r}{R}\cos(\tau Z)\boldsymbol{e}_\Theta,$$

$$\boldsymbol{e}_z = \boldsymbol{e}_Z.$$

Therefore, we have

$$[\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_i] = \begin{bmatrix} \cos(\tau Z) & -r\sin(\tau Z) & 0 \\ \frac{1}{R}\sin(\tau Z) & \frac{r}{R}\cos(\tau Z) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$[F^\alpha_\beta] = \begin{bmatrix} \sqrt{a}\cos(\tau Z) & -r\sin(\tau Z) & -\tau r\sin(\tau Z) \\ \sqrt{a}\frac{1}{R}\sin(\tau Z) & \frac{r}{R}\cos(\tau Z) & \tau\frac{r}{R}\cos(\tau Z) \\ 0 & 0 & 1/a \end{bmatrix}.$$

One can easily check that $\det[F^\alpha_\beta] = 1$ and that $[C_{\alpha\beta}] = [F^\gamma_\alpha]^T[g_{\gamma\delta}][F^\delta_\beta]$ agrees with the previous result.

One can similarly obtain the matrix $[F^i{}_j] = [F^i{}_\alpha][\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_j]$,

$$[F^i{}_j] = \begin{bmatrix} \sqrt{a}\cos(\tau Z) & -\sqrt{a}\,r\sin(\tau Z) & 0 \\ \frac{1}{R}\sin(\tau Z) & \frac{r}{R}\cos(\tau Z) & \tau \\ 0 & 0 & 1/a \end{bmatrix}.$$

Exercise 1.3.1

1) $\boldsymbol{u} = (x - X,\, y - Y,\, z - Z) = (\kappa Y, 0, 0) = \kappa Y \boldsymbol{e}_x$.

2) $H = \nabla_{\boldsymbol{X}} \boldsymbol{u} = \kappa\, \boldsymbol{e}_x \otimes \boldsymbol{e}_y$, hence, in components relative to the standard basis,

$$\widetilde{E} = \begin{bmatrix} 0 & \kappa/2 & 0 \\ \kappa/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \widetilde{R} = \begin{bmatrix} 0 & \kappa/2 & 0 \\ -\kappa/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3) For $\kappa \ll 1$, since $\theta = \tan^{-1}(\kappa/2) \approx \kappa/2$, $\sin\theta \approx 0$, and $\cos\theta \approx 1$, one can easily check the relation (1.22) with the results for finite shear.

Exercise 1.3.2

1) $[E_{\langle\alpha\beta\rangle}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau r/2 \\ 0 & \tau r/2 & \tau^2 r^2/2 \end{bmatrix}.$

2) $[\widetilde{E}_{\langle\alpha\beta\rangle}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau r/2 \\ 0 & \tau r/2 & 0 \end{bmatrix}, \qquad [\widetilde{R}_{\langle\alpha\beta\rangle}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau r/2 \\ 0 & -\tau r/2 & 0 \end{bmatrix}.$

Exercise 1.4.1

Let $\boldsymbol{a}$ be a constant vector, then

$$\begin{aligned}
\boldsymbol{a} \cdot \mathrm{Div}(JF^{-T}) &\overset{(A.75)}{=} \mathrm{Div}(JF^{-1}\boldsymbol{a}) \\
&\overset{(A.76)_1}{=} F^{-1}\boldsymbol{a} \cdot \mathrm{Grad}\, J + J\, \mathrm{Div}(F^{-1}\boldsymbol{a}) \\
&\overset{(1.33)_1}{=} F^{-1}\boldsymbol{a} \cdot J\, \mathrm{div}\, F^T + J\, \mathrm{tr}\, \mathrm{Grad}(F^{-1}\boldsymbol{a}) \\
&\overset{(1.30)_2}{=} J(F^{-1}\boldsymbol{a} \cdot \mathrm{div}\, F^T + \mathrm{tr}(F\, \mathrm{grad}(F^{-1}\boldsymbol{a})) \\
&\overset{(A.76)_2}{=} J\, \mathrm{div}(F(F^{-1}\boldsymbol{a})) = J\, \mathrm{div}\, \boldsymbol{a} = 0,
\end{aligned}$$

which proves $(1.34)_1$. Furthermore, from

$$\mathrm{Div}(JF^{-T}) = J\, \mathrm{Div}(F^{-T}) + F^{-T}\, \mathrm{Grad}\, J \overset{(1.30)_1}{=} J\, \mathrm{Div}(F^{-T}) + \mathrm{grad}\, J,$$

$(1.34)_1$ implies $(1.34)_2$.

Exercise 1.4.2

Use Exercise A.2.11.

Exercise 1.4.3

1) Note that $\dot{Q}(\boldsymbol{X}-\boldsymbol{X}_\circ) = \dot{Q}Q^T(\boldsymbol{x}-\boldsymbol{x}_\circ) = -Q\dot{Q}^T(\boldsymbol{x}-\boldsymbol{x}_\circ) = \boldsymbol{\omega}\times(\boldsymbol{x}-\boldsymbol{x}_\circ)$, by $\boldsymbol{\omega} = \langle Q\dot{Q}^T\rangle$ and (1) of Exercise A.1.6.
2) For this motion, $F = Q(t) \in \mathcal{O}(V)$, therefore, it preserves the inner product, and by (1.8), $d\boldsymbol{x}_1 \cdot d\boldsymbol{x}_2 = d\boldsymbol{X}_1 \cdot d\boldsymbol{X}_2$.

Exercise 1.4.4

1) Since $\boldsymbol{a} = (\ddot{x}, \ddot{y}) = (k^2x, k^2y)$, we have

   $\ddot{x} = k^2x, \qquad \ddot{y} = k^2y.$

   With the given initial conditions, the solutions are

   $x = Xe^{kt}, \qquad y = Ye^{-kt},$

   and the deformation can be written as

   $\boldsymbol{x} = \chi_\kappa(X, Y, t) = Xe^{kt}\boldsymbol{e}_x + Ye^{-kt}\boldsymbol{e}_y.$

2) $\dot{x} = kXe^{kt} = kx,$

   $\dot{y} = -kYe^{-kt} = -ky.$

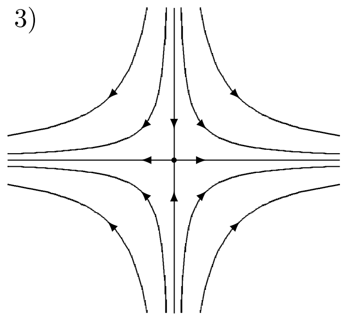
   Therefore, in material description,

   $\boldsymbol{v}(X, Y, t) = kXe^{kt}\boldsymbol{e}_x - kYe^{-kt}\boldsymbol{e}_y,$

   and in spatial description,

   $\boldsymbol{v}(x, y, t) = kx\,\boldsymbol{e}_x - ky\,\boldsymbol{e}_y.$

3)



Exercise 1.5.1

1) From $\boldsymbol{v} = (\dot{x}, \dot{y}) = (u(y), 0)$, we have $\dot{x} = u(y)$, $\dot{y} = 0$. With the initial conditions: $x(X, Y, 0) = X$, $y(X, Y, 0) = Y$, the solutions are

   $x(X, Y, t) = X + u(Y)t, \quad y(X, Y, t) = Y.$

   Therefore, we have

   $\boldsymbol{x} = \chi_t(X, Y) = (X + u(Y)t)\boldsymbol{e}_x + Y\boldsymbol{e}_y,$

   $\boldsymbol{X} = \chi_t^{-1}(x, y) = (x - u(y)t)\boldsymbol{e}_x + y\boldsymbol{e}_y.$

   From (1.38), the relative deformation is given by

   $$\begin{aligned}\boldsymbol{\xi} &= \chi_{(t)}(x, y, \tau) = \chi_\tau(\chi_t^{-1}(x, y)) \\ &= \chi_\tau\big((x - u(y)t)\boldsymbol{e}_x + y\boldsymbol{e}_y\big) = (x - u(y)(t-\tau))\boldsymbol{e}_x + y\boldsymbol{e}_y.\end{aligned}$$

2) $F_t(x,y,\tau) = \boldsymbol{e}_x \otimes \boldsymbol{e}_x + \kappa(\tau - t)\boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_y = \mathit{1} + (\tau - t)\kappa\, \boldsymbol{e}_x \otimes \boldsymbol{e}_y$,
since $\boldsymbol{e}_x \otimes \boldsymbol{e}_x + \boldsymbol{e}_y \otimes \boldsymbol{e}_y = \mathit{1}$.

Exercise 1.5.2

We obtain $\boldsymbol{v} = (2tXe^{t^2},\, Ye^t,\, 0) = (2tx,\, y,\, 0)$, therefore, the streamline, passing through $\boldsymbol{x}_\circ$ at time t, satisfies

$\dot{x}(s) = 2t\, x(s), \quad \dot{y}(s) = y(s), \quad \dot{z}(s) = 0,$

with the initial conditions: $x(0) = x_\circ,\ y(0) = y_\circ,\ z(0) = z_\circ$.

The solution curve is given by

$\boldsymbol{x}(s) = (x_\circ e^{2ts},\, y_\circ e^s,\, z_\circ)$.

Exercise 1.6.1

Using (1.31), the decomposition $F = RU$, and the fact that $\dot{R}R^T$ is skew-symmetric, we have

$L = \dot{F}F^{-1} = (\dot{R}U + R\dot{U})(RU)^{-1} = \dot{R}R^T + R\dot{U}U^{-1}R^T,$

whose skew-symmetric part is

$W = \dot{R}R^T + \frac{1}{2}R(\dot{U}U^{-1} - U^{-1}\dot{U})R^T.$

(Erratum: This is the correct formula for W).

Exercise 1.6.2

The results follow immediately from the relations:

$\dot{\boldsymbol{v}} = \dfrac{\partial \boldsymbol{v}}{\partial t} + (\operatorname{grad} \boldsymbol{v})\boldsymbol{v}$, by (1.32);

$\operatorname{grad}(\boldsymbol{v} \cdot \boldsymbol{v}) = 2(\operatorname{grad} \boldsymbol{v})^T \boldsymbol{v}$, by (A.52);

$\boldsymbol{w} = \langle -2W \rangle$, and (1) of Exercise A.1.6.

Exercise 1.6.3

From Exercise 1.5.1,

$$C_t(\tau) = F_t(\tau)^T F_t(\tau) = (\mathit{1} + (\tau - t)\kappa N)^T(\mathit{1} + (\tau - t)\kappa N)$$
$$= \mathit{1} + (\tau - t)\kappa(N + N^T) + (\tau - t)^2\kappa^2 N^T N.$$

Exercise 1.7.1

From (1.57), we obtain

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} + (\dot{Q}_{jk}(x_k - x_k^\circ) + \dot{c}_j)\frac{\partial}{\partial x_j^*} \overset{(1.67)}{=} \frac{\partial}{\partial t^*} + (\Omega_{jk}(x_k^* - c_k) + \dot{c}_j)\frac{\partial}{\partial x_j^*}\,,$$

$$\frac{\partial}{\partial x_i} = Q_{ji}\frac{\partial}{\partial x_j^*}\,.$$

Use of (1.66) completes the proof.

Exercise 1.7.2

By assumption, $S^* = QSQ^T$,

1) $$\frac{\partial S^*_{ij}}{\partial x^*_k} = Q_{ip}Q_{jq}\frac{\partial S_{pq}}{\partial x_r}\frac{\partial x_r}{\partial x^*_k} = Q_{ip}Q_{jq}Q_{kr}\frac{\partial S_{pq}}{\partial x_r},$$

which shows that $\operatorname{grad} S$ is an objective third order tensor.

2) $$\dot{S}^* = \dot{Q}SQ^T + Q\dot{S}Q^T + QS\dot{Q}^T \overset{(1.67)}{=} Q\dot{S}Q^T + \Omega(QSQ^T) - (QSQ^T)\Omega,$$

therefore, $\dot{S}$ is not objective.

Exercise 1.7.3

Note that $P(t+h) = R_t(t+h) = 1 + h\,W(t) + o(2)$, hence,

$$\overset{\circ}{S}(t) = \lim_{h\to 0}\frac{1}{h}\big(S(t+h) - (1 + h\,W(t))S(t)(1 + h\,W(t)^T)\big) = \dot{S} - WS + SW.$$

To show it is objective, use the result (2) of the preceding Exercise and the relation $(1.78)_2$.

Exercise 1.7.4

Note that $P(t+h) = F_t(t+h) = 1 + h\,L(t) + o(2)$ by (1.46).

Exercise 1.7.5

1) We have from (1.42),

$$F^*_{t^*}(\tau^*) = F^*(\tau^*)F^*(t^*)^{-1}$$
$$\overset{(1.74)}{=} Q(\tau)F(\tau)K^T(Q(t)F(t)K^T)^{-1} = Q(\tau)F_t(\tau)Q^T(t).$$

By polar decomposition,

$$R^*_{t^*}(\tau^*)U^*_{t^*}(\tau^*) = Q(\tau)R_t(\tau)U_t(\tau)Q^T(t)$$
$$= Q(\tau)R_t(\tau)Q^T(t)Q(t)U_t(\tau)Q^T(t),$$

which by uniqueness of the decomposition, implies

$$U^*_{t^*}(\tau^*) = Q(t)U_t(\tau)Q^T(t),$$
$$R^*_{t^*}(\tau^*) = Q(\tau)R_t(\tau)Q^T(t).$$

Therefore, U_t is objective and R_t is not.

2) From (1), we also have $C^*_{t^*}(\tau^*) = Q(t)C_t(\tau)Q^T(t)$, or, C_t is objective. Since by $(1.57)_2$, $\tau^* = \tau + a$, by taking n-th derivative with respect to τ, and using the definition (1.53), it follows that A_n is objective.

Exercise 2.1.1

1) Since the gradient of $f(\boldsymbol{X}, t)$ is normal to the surface, therefore, the unit normal of the surface $\mathcal{S}_\kappa$ is given by

$$\boldsymbol{n}_\kappa = \frac{\operatorname{Grad} f}{|\operatorname{Grad} f|}\,.$$

The normal speed of $\mathcal{S}_\kappa(t)$ can be defined as $U_\kappa = \dot{\mathcal{S}}_\kappa(t)\cdot\boldsymbol{n}_\kappa$, and since $\boldsymbol{X} = \mathcal{S}_\kappa(t)$, from

$$\frac{d}{dt} f(\mathcal{S}_\kappa(t), t) = \dot{f} + \operatorname{Grad} f \cdot \dot{\mathcal{S}}_\kappa(t) = 0,$$

we obtain

$$U_\kappa = -\frac{\dot{f}}{|\operatorname{Grad} f|}\,.$$

Likewise, we can prove the similar relations in spatial configuration, by writing the moving surface as $f(\boldsymbol{x}, t) = \tilde{f}(\chi_\kappa(\boldsymbol{X}, t), t) = f(\boldsymbol{X}, t)$, and $\boldsymbol{x} = \mathcal{S}(t)$ and defining the normal speed as $u_n = \dot{\mathcal{S}}(t)\cdot\boldsymbol{n}$.

2) Since $\operatorname{Grad} f = F^T \operatorname{grad} f$, from which we obtain, by (1),

$$\boldsymbol{n}_\kappa = \frac{F^T\boldsymbol{n}}{|F^T\boldsymbol{n}|}, \quad \text{and} \quad |F^T\boldsymbol{n}| = \frac{|\operatorname{Grad} f|}{|\operatorname{grad} f|}\,.$$

Similarly, $\dot{f} = \dfrac{\partial f}{\partial t} + \operatorname{grad} f \cdot \dot{\boldsymbol{x}}$, from which, by (1) and (2.17), we obtain

$$U_\kappa = \frac{U}{|F^T\boldsymbol{n}|}\,.$$

Exercise 2.2.1

Note that $\operatorname{div}(\rho\psi\otimes\boldsymbol{v}) = \psi\operatorname{div}(\rho\boldsymbol{v}) + (\operatorname{grad}\psi)(\rho\boldsymbol{v})$, and when ψ is a scalar quantity, this relation should read, $\operatorname{div}(\rho\psi\boldsymbol{v}) = \psi\operatorname{div}(\rho\boldsymbol{v}) + (\operatorname{grad}\psi)\cdot(\rho\boldsymbol{v})$.

Exercise 2.2.2

Substitute ψ for $\rho\psi$ in (2.4), and use the relation (2.36).

Exercise 2.3.1

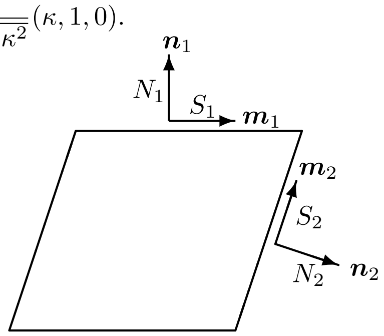
From (1.14), we have

$$T = -p\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \mu\begin{bmatrix} 1+\kappa^2 & \kappa & 0\\ \kappa & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

From the figure, we denote the unit normals and the unit tangents by

$$\boldsymbol{n}_1 = (0,1,0), \quad \boldsymbol{m}_1 = (1,0,0),$$

$$\boldsymbol{n}_2 = \frac{1}{\sqrt{1+\kappa^2}}(1,-\kappa,0), \quad \boldsymbol{m}_2 = \frac{1}{\sqrt{1+\kappa^2}}(\kappa,1,0).$$



Then the normal and tangential components are

$$N_1 = \boldsymbol{n}_1 \cdot T\boldsymbol{n}_1 = -p + \mu,$$

$$S_1 = \boldsymbol{m}_1 \cdot T\boldsymbol{n}_1 = \mu\kappa,$$

$$N_2 = \boldsymbol{n}_2 \cdot T\boldsymbol{n}_2 = -p + \frac{\mu}{1+\kappa^2},$$

$$S_2 = \boldsymbol{m}_2 \cdot T\boldsymbol{n}_2 = \frac{\mu\kappa}{1+\kappa^2}.$$

Exercise 2.3.2

By (1.52), (1.55), and the assumptions, the equation (2.57) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\,\mathrm{grad}(\boldsymbol{v}\cdot\boldsymbol{v}) + \mathrm{curl}\,\boldsymbol{v}\times\boldsymbol{v} + \frac{1}{\rho}\,\mathrm{grad}\,p + \mathrm{grad}\,\phi = 0.$$

Exercise 2.5.1

Note that from (2.17), $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!] = -[\![U]\!]$, because u_n is continuous at the singular surface. Taking inner product of $(2.80)_2$ with $\boldsymbol{n}$, the first relation of (2.81) follows immediately. From $(2.80)_2$, we also obtain

$$\rho U[\![\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}]\!] = [\![p]\!]\boldsymbol{n} - \rho U[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]\boldsymbol{n} = 0.$$

For the proof of the remaining relation, see Exercise 2.5.2.

Exercise 2.5.2

Let $\boldsymbol{v} = \boldsymbol{v}_n + \boldsymbol{v}_t$ be the decomposition of $\boldsymbol{v}$ into normal and tangential velocities. Then, $[\![\boldsymbol{v}_t]\!] = 0$ by assumption, and we have

$$\frac{m}{2}[\![v^2]\!] = \frac{m}{2}[\![(\boldsymbol{v}_n+\boldsymbol{v}_t)\cdot(\boldsymbol{v}_n+\boldsymbol{v}_t)]\!] = \frac{m}{2}[\![(\boldsymbol{v}.\boldsymbol{n})^2]\!] + m[\![\boldsymbol{v}_n]\!]\cdot\boldsymbol{v}_t,$$

and

$$[\![T\boldsymbol{v}]\!]\cdot\boldsymbol{n} = [\![\boldsymbol{v}_n\cdot T\boldsymbol{n}]\!] + \boldsymbol{v}_t\cdot[\![T\boldsymbol{n}]\!] = [\![(\boldsymbol{v}_n - u_n\boldsymbol{n})\cdot T\boldsymbol{n}]\!] + u_n[\![\boldsymbol{n}\cdot T\boldsymbol{n}]\!] + \boldsymbol{v}_t\cdot[\![T\boldsymbol{n}]\!]$$

$$\overset{(2.80)}{=} [\![(\boldsymbol{v}\cdot\boldsymbol{n} - u_n)\boldsymbol{n}\cdot T\boldsymbol{n}]\!] - m u_n[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!] - m\boldsymbol{v}_t\cdot[\![\boldsymbol{v}_n]\!]$$

$$\overset{(2.17)}{=} -m[\![\frac{1}{\rho}(\boldsymbol{n}\cdot T\boldsymbol{n})]\!] - m u_n[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!] - m\boldsymbol{v}_t\cdot[\![\boldsymbol{v}_n]\!].$$

From the above relations and $[\![(\boldsymbol{v}\cdot\boldsymbol{n})^2 - 2(\boldsymbol{v}\cdot\boldsymbol{n})u_n]\!] = [\![U^2]\!]$, because $[\![u_n]\!] = 0$, the jump condition $(2.80)_3$ leads to (2.82).

Exercise 2.5.3

For a moving singular surface, the first two relations are trivial consequences of $(2.87)_{1,2}$ by (2.26). To verify the third relation, we shall use the following identity,

$$[\![AB]\!] = [\![A]\!]\langle B\rangle + \langle A\rangle[\![B]\!],$$

which can easily be verified, and note that

$$\frac{1}{2}\,\rho_\kappa U_\kappa[\![\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}]\!] = \rho_\kappa U_\kappa[\![\dot{\boldsymbol{x}}]\!]\cdot\langle\dot{\boldsymbol{x}}\rangle \overset{(2.26)}{=} -\rho_\kappa U_\kappa^2[\![F\boldsymbol{n}_\kappa]\!]\cdot\langle\dot{\boldsymbol{x}}\rangle$$

$$\overset{(2.88)_2}{=} -[\![T_\kappa\boldsymbol{n}_\kappa]\!]\cdot\langle\dot{\boldsymbol{x}}\rangle = \langle T_\kappa\boldsymbol{n}_\kappa\rangle\cdot[\![\dot{\boldsymbol{x}}]\!] - [\![T_\kappa\boldsymbol{n}_\kappa\cdot\dot{\boldsymbol{x}}]\!].$$

Therefore, by (2.26) again, we obtain

$$\frac{1}{2}\,\rho_\kappa U_\kappa[\![\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}]\!] + [\![T_\kappa^T\dot{\boldsymbol{x}}]\!]\cdot\boldsymbol{n}_\kappa = -U_\kappa\,\langle T_\kappa\boldsymbol{n}_\kappa\rangle\cdot[\![F\boldsymbol{n}_\kappa]\!],$$

from which the result follows immediately.

Exercise 3.2.1

1) Note that if $AB = BA$, then

$$(A+B)^n = \sum_{k=0}^{n}\binom{n}{k}A^kB^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

and the proof follows from the following identity for the product of two infinite series,

$$\Big(\sum_{i=0}^{\infty}a_i\Big)\Big(\sum_{j=0}^{\infty}b_j\Big) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\Big(a_kb_{n-k}\Big).$$

2) It is easy to verify that $(\exp(A))^T = \exp(A^T)$.

3) Verify that $\dfrac{d}{dt}\big(\exp(tA)\big) = \big(\exp(tA)\big)A$.

Exercise 3.2.2

Given the constitutive equation,

$$T_{ij}(D_{kl}) = -p\delta_{ij} + V_{ijkl}D_{kl}.$$

from the material objectivity condition,

$T_{ij}(Q_{km}Q_{ln}D_{mn}) = Q_{ip}Q_{jq}T_{pq}(D_{kl})$, we obtain

$$V_{ijkl}Q_{km}Q_{ln}D_{mn} = Q_{ip}Q_{jq}V_{pqkl}D_{kl},$$

and it reduces to

$$V_{ijkl}Q_{km}Q_{ln} = Q_{ip}Q_{jq}V_{pqmn},$$

which multiplied by $Q_{rm}Q_{sn}$ becomes (3.24).

Exercise 3.3.1

By assumption, $Q = \nabla_{\boldsymbol{X}}(\hat{\kappa} \circ \kappa^{-1})$ is orthogonal, and we have

$\mathcal{T}_{\hat{\kappa}}(F) = \mathcal{T}_{\kappa}(FQ) \quad \forall\, F,$

In particular, for $F = \mathit{1}$, we have, by the condition of material objectivity,

$\mathcal{T}_{\hat{\kappa}}(\mathit{1}) = \mathcal{T}_{\kappa}(Q) = Q\mathcal{T}_{\kappa}(\mathit{1})Q^T,$

from which, $\mathcal{T}_{\kappa}(\mathit{1}) = 0$ implies $\mathcal{T}_{\hat{\kappa}}(\mathit{1}) = 0$ also.

Exercise 3.3.2

If σ satisfies objectivity condition, then $\sigma(F) = \sigma(QF)$ for any Q orthogonal. Taking the gradient with respect to F by the chain rule, we obtain, for any S,

$(\partial_F \sigma)|_F [S] = (\partial_F \sigma)|_{QF} [\partial_F (QF)[S]] = (\partial_F \sigma)|_{QF} [QS] = Q^T (\partial_F \sigma)|_{QF} [S],$

which implies that $(\partial_F \sigma)|_{QF} = Q\,(\partial_F \sigma)|_F$. Therefore,

$\mathcal{T}(QF) = \rho(\partial_F \sigma)|_{QF} (QF)^T = \rho\, Q(\partial_F \sigma)|_F F^T Q^T = Q\mathcal{T}(F)Q^T.$

Exercise 3.3.3

The function $\widetilde{\mathcal{F}}_{\kappa}$ and $\mathcal{F}_{\kappa}$ are related by

$\widetilde{\mathcal{F}}_{\kappa}(F^t, \theta^t, \boldsymbol{g}^t) = \widetilde{\mathcal{F}}_{\kappa}(F^t, \theta^t, (F^{-T})^t \boldsymbol{g}^t_{\kappa}) = \mathcal{F}_{\kappa}(F^t, \theta^t, \boldsymbol{g}^t_{\kappa}).$

Therefore,

$\mathcal{F}_{\kappa}(Q^t F^t, \theta^t, \boldsymbol{g}^t_{\kappa}) = \widetilde{\mathcal{F}}_{\kappa}(Q^t F^t, \theta^t, Q^t (F^{-T})^t \boldsymbol{g}^t_{\kappa}) = \widetilde{\mathcal{F}}_{\kappa}(Q^t F^t, \theta^t, Q^t \boldsymbol{g}^t).$

Exercise 3.4.1

Since $B = FF^T$ and $T = \mathcal{T}(F) = t_0(B)\mathit{1} + t_1(B)B + t_2(B)B^2$, where $t_i(B) = t_i(\operatorname{tr} B, \operatorname{tr} B^2, \operatorname{tr} B^3)$, we have

$\mathcal{T}(QF) = t_0(QBQ^T)\mathit{1} + t_1(QBQ^T)QBQ^T + t_2(QBQ^T)QB^2Q^T$

and $t_i(QBQ^T) = t_i(B)$. Therefore, it satisfies the objectivity condition,

$\mathcal{T}(QF) = Q\mathcal{T}(F)Q^T.$

To show that it is a special case of (3.35), simply observe that, by the polar decomposition $F = RU$, $B = R(U^2)R^T$.

Exercise 3.4.2

From (3.35), $\mathcal{T}(F^t) = R\mathcal{T}(U^t)R^T \overset{(1.13)}{=} R\mathcal{T}(\sqrt{C^t})R^T$. On the other hand, from (1.42), we have $F^t(s) = F^t_t(s)F(t)$, and hence

$C^t = (F^t)^T F^t = F^T C^t_t F = UR^T C^t_t RU = \sqrt{C}(R^T C^t_t R)\sqrt{C}.$

Therefore, we can write $\mathcal{T}(\sqrt{C^t}) = \widehat{\mathcal{T}}(R^T C^t_t R;\, C)$.

Moreover, it is straightforward to prove that $\mathcal{T}(F^t) = R\widehat{\mathcal{T}}(R^T C^t_t R;\, C)R^T$ satisfies the objectivity condition.

Exercise 3.5.1

Consider the stress tensor given by $T = \mathcal{T}_\kappa(F)$, then if $G \in \mathcal{G}_\kappa(T)$, $\mathcal{T}_\kappa(FG) = \mathcal{T}_\kappa(F)$. On the other hand, by the objectivity condition, we have, $\mathcal{T}_\kappa(QFG) = Q\mathcal{T}_\kappa(FG)Q^T$ for any orthogonal tensor Q. In particular, for $Q = -\mathit{1}$, we obtain

$$\mathcal{T}_\kappa(F(-G)) = \mathcal{T}_\kappa(-FG) = \mathcal{T}_\kappa(FG) = \mathcal{T}_\kappa(F).$$

Therefore, by definition, $-G \in \mathcal{G}_\kappa(T)$ also.

(Erratum: The original expression $T = \mathcal{T}_\kappa(F, \theta, \boldsymbol{g})$ should read $T = \mathcal{T}_\kappa(F)$. Note that one can also show that the above conclusion does not hold when the constitutive function also depends on the temperature gradient.)

Exercise 3.5.2

Use the Noll's rule (3.44).

Exercise 3.5.3

Since κ and $\hat{\kappa}$ are undistorted configurations of a solid, both $\mathcal{G}_\kappa$ and $\mathcal{G}_{\hat{\kappa}}$ are subgroups of $\mathcal{O}(V)$. If $Q \in \mathcal{G}_\kappa$, by Noll's rule, $PQP^{-1} \in \mathcal{G}_{\hat{\kappa}}$, and $PQP^{-1} = RUQU^{-1}R^T$ by the decomposition $P = RU$, therefore, UQU^{-1} is an orthogonal tensor, say G, and hence, we have

$$GU = UQ = Q(Q^TUQ).$$

Since (Q^TUQ) is symmetric positive definite, by the uniqueness of polar decomposition, it follows that $U = Q^TUQ$.

Moreover, since $QU = UQ$, we have

$$RQR^T = RUU^{-1}QUU^{-1}R^T = RUQ(RU)^{-1} = PQP^{-1},$$

which implies that $RQR^T \in \mathcal{G}_{\hat{\kappa}}$ by the Noll's rule.

Exercise 3.5.4

Let $P = \{\alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2,\ \alpha_1, \alpha_2 \in I\!R\}$ be the plane perpendicular to $\boldsymbol{e}_3$ and $\boldsymbol{v} \in P$. Then $G\boldsymbol{v} \in P$ for any $G \in \mathcal{G}_\kappa$, i.e., the material has a characteristic plane P preserved by the group $\mathcal{G}_\kappa$.

Exercise 3.6.1

By assumption, $\mathcal{G}_\kappa = \mathcal{O}(V)$, and we have

$$\mathcal{T}(F, \theta, \boldsymbol{g}) = \mathcal{T}(FQ, \theta, \boldsymbol{g}) = \mathcal{T}(VRQ, \theta, \boldsymbol{g}) \quad \forall\, Q \in \mathcal{O}(V),$$

by the polar decomposition $F = VR$. Taking $Q = R^T$, we obtain

$$\mathcal{T}(F, \theta, \boldsymbol{g}) = \mathcal{T}(V, \theta, \boldsymbol{g}) = \widehat{\mathcal{T}}(B, \theta, \boldsymbol{g}).$$

Moreover, we have

$$\mathcal{T}(QF, \theta, Q\boldsymbol{g}) = \widehat{\mathcal{T}}((QF)(QF)^T, \theta, Q\boldsymbol{g}) = \widehat{\mathcal{T}}(QBQ^T, \theta, Q\boldsymbol{g}),$$

which implies that $\widehat{\mathcal{T}}(B, \theta, \boldsymbol{g})$ is an isotropic symmetric tensor-valued function, by the objectivity condition of Exercise 3.3.3.

Exercise 4.2.2

If $S(\boldsymbol{v}, \boldsymbol{u})$ is defined by $S_{ij} = M_{ijk}u_k$, then S is a second-order tensor isotropic function. Since it is linear in $\boldsymbol{u}$, we have the following representation,

$$S(\boldsymbol{v}, \boldsymbol{u}) = f_1 \boldsymbol{v} \otimes \boldsymbol{u} + f_2 \boldsymbol{u} \otimes \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{u})(f_3 \mathit{1} + f_4 \boldsymbol{v} \otimes \boldsymbol{v}),$$

where f_1, f_2, f_3, and f_4 are functions of $(\boldsymbol{v} \cdot \boldsymbol{v})$. Therefore, we obtain

$$M_{ijk}(\boldsymbol{v}) = f_1 v_i \delta_{jk} + f_2 v_j \delta_{ki} + f_3 v_k \delta_{ji} + f_4 v_i v_j v_k.$$

Exercise 4.2.3

1) $S(QAQ^T) = (\mathit{1} + QAQ^T)^{-1} = (Q(\mathit{1} + A)Q^T)^{-1} = QS(A)Q^T.$
2) By Theorem 4.2.2, we have the representation,

$$(\mathit{1} + A)^{-1} = s_0 \mathit{1} + s_1 A + s_2 A^2.$$

Multiplying by $(\mathit{1} + A)$, we obtain

$$s_0 \mathit{1} + (s_0 + s_1)A + (s_1 + s_2)A^2 + s_2 A^3 = \mathit{1},$$

which, by the use of Cayley-Hamilton theorem, becomes

$$(s_0 + s_2 I\!I\!I_A)\mathit{1} + (s_0 + s_1 - s_2 I\!I_A)A + (s_1 + s_2 + s_2 I_A)A^2 = \mathit{1}.$$

Comparing coefficients on both sides, we obtain

$$s_0 + s_2 I\!I\!I_A = 1, \quad s_0 + s_1 - s_2 I\!I_A = 0, \quad s_1 + s_2 + s_2 I_A = 0,$$

which lead to the expressions of the problem.

Exercise 4.3.1

1) $\phi(\boldsymbol{u}, \boldsymbol{v}, A) = f(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{u} \cdot \boldsymbol{v}, \operatorname{tr} A, \operatorname{tr} A^2, \operatorname{tr} A^3, \boldsymbol{u} \cdot A\boldsymbol{u}, \boldsymbol{v} \cdot A\boldsymbol{v}, \boldsymbol{u} \cdot A\boldsymbol{v}, \boldsymbol{u} \cdot A^2\boldsymbol{u}, \boldsymbol{v} \cdot A^2\boldsymbol{v}, \boldsymbol{u} \cdot A^2\boldsymbol{v}),$

$$\boldsymbol{h}(\boldsymbol{u}, \boldsymbol{v}, A) = h_1 \boldsymbol{u} + h_2 A\boldsymbol{u} + h_3 A^2\boldsymbol{u} + h_4 \boldsymbol{v} + h_5 A\boldsymbol{v} + h_6 A^2\boldsymbol{v},$$

$$\begin{aligned} S(\boldsymbol{u}, \boldsymbol{v}, A) &= s_1 \mathit{1} + s_2 A + s_3 A^2 + s_4 \boldsymbol{u} \otimes \boldsymbol{u} + s_5 \boldsymbol{v} \otimes \boldsymbol{v} \\ &\quad + s_6(\boldsymbol{u} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{u}) + s_7(\boldsymbol{u} \otimes A\boldsymbol{u} + A\boldsymbol{u} \otimes \boldsymbol{u}) + s_8(\boldsymbol{v} \otimes A\boldsymbol{v} + A\boldsymbol{v} \otimes \boldsymbol{v}) \\ &\quad + s_9 A\boldsymbol{u} \otimes A\boldsymbol{u} + s_{10} A\boldsymbol{v} \otimes A\boldsymbol{v} + s_{11}(\boldsymbol{u} \otimes A\boldsymbol{v} + A\boldsymbol{v} \otimes \boldsymbol{u}) \\ &\quad + s_{12}(\boldsymbol{v} \otimes A\boldsymbol{u} + A\boldsymbol{u} \otimes \boldsymbol{v}), \end{aligned}$$

where the coefficients are isotropic scalar functions of $(\boldsymbol{u}, \boldsymbol{v}, A)$.

2) $\phi(\boldsymbol{u},\boldsymbol{v},A) = \phi_0 + \phi_1 \boldsymbol{u}\cdot\boldsymbol{v} + \phi_2 \operatorname{tr} A + \phi_3 \boldsymbol{u}\cdot A\boldsymbol{u}$,

$$\boldsymbol{h}(\boldsymbol{u},\boldsymbol{v},A) = (\alpha_0 + \alpha_1 \boldsymbol{u}\cdot\boldsymbol{v} + \alpha_2 \operatorname{tr} A + \alpha_3 \boldsymbol{u}\cdot A\boldsymbol{u})\boldsymbol{u} + \eta_1\boldsymbol{v} + \eta_2 A\boldsymbol{u},$$

$$\begin{aligned} S(\boldsymbol{u},\boldsymbol{v},A) &= (\beta_0 + \beta_1 \boldsymbol{u}\cdot\boldsymbol{v} + \beta_2 \operatorname{tr} A + \beta_3 \boldsymbol{u}\cdot A\boldsymbol{u})\mathit{1} \\ &+ (\gamma_0 + \gamma_1 \boldsymbol{u}\cdot\boldsymbol{v} + \gamma_2 \operatorname{tr} A + \gamma_3 \boldsymbol{u}\cdot A\boldsymbol{u})\boldsymbol{u}\otimes\boldsymbol{u} \\ &+ \sigma_1 A + \sigma_2(\boldsymbol{u}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{u}) + \sigma_3(\boldsymbol{u}\otimes A\boldsymbol{u} + A\boldsymbol{u}\otimes\boldsymbol{u}), \end{aligned}$$

where the coefficients are functions of $(\boldsymbol{u}\cdot\boldsymbol{u})$ only.

Exercise 4.3.2

1) Let $H = \partial_{\boldsymbol{v}}\boldsymbol{h}$, then by definition, $H(A,\boldsymbol{v})[\boldsymbol{u}] = \dfrac{d}{dt}\boldsymbol{h}(A,\boldsymbol{v}+t\boldsymbol{u})\Big|_{t=0}$. Therefore, for any orthogonal tensor Q, we have

$$\begin{aligned} H(QAQ^T, Q\boldsymbol{v})[\boldsymbol{u}] &= \frac{d}{dt}\boldsymbol{h}(QAQ^T, Q\boldsymbol{v}+t\boldsymbol{u})\Big|_{t=0} \\ &= \frac{d}{dt}\boldsymbol{h}(QAQ^T, Q(\boldsymbol{v}+tQ^T\boldsymbol{u}))\Big|_{t=0} = \frac{d}{dt}\Big(Q\boldsymbol{h}(A,\boldsymbol{v}+tQ^T\boldsymbol{u})\Big)\Big|_{t=0}, \end{aligned}$$

since $\boldsymbol{h}(A,\boldsymbol{v})$ is isotropic. Then by definition again, we obtain

$$H(QAQ^T, Q\boldsymbol{v})[\boldsymbol{u}] = QH(A,\boldsymbol{v})[Q^T\boldsymbol{u}] = QH(A,\boldsymbol{v})Q^T[\boldsymbol{u}],$$

which implies that $H(QAQ^T, Q\boldsymbol{v}) = QH(A,\boldsymbol{v})Q^T$. Therefore, $\partial_{\boldsymbol{v}}\boldsymbol{h}$ is an isotropic function.

2) We have the representation, $\boldsymbol{h}(A,\boldsymbol{v}) = g_1\boldsymbol{v} + g_2A\boldsymbol{v} + g_3A^2\boldsymbol{v}$, where $g_a = g_a(\operatorname{tr} A, \operatorname{tr} A^2, \operatorname{tr} A^3, I_1, I_2, I_3)$ and $I_1 = \boldsymbol{v}\cdot\boldsymbol{v}$, $I_2 = \boldsymbol{v}\cdot A\boldsymbol{v}$, $I_3 = \boldsymbol{v}\cdot A^2\boldsymbol{v}$. We obtain the gradient $\partial_{\boldsymbol{v}}\boldsymbol{h}$ explicitly in terms of Cartesian components,

$$\begin{aligned} \frac{\partial h_i}{\partial v_j} &= g_1\delta_{ij} + g_2A_{ij} + g_3A^2_{ij} \\ &+ 2\frac{\partial g_1}{\partial I_1}v_iv_j + \frac{\partial g_1}{\partial I_2}v_i(A_{jl}+A_{lj})v_l + \frac{\partial g_1}{\partial I_3}v_i(A^2_{jl}+A^2_{lj})v_l \\ &+ 2\frac{\partial g_2}{\partial I_1}A_{ik}v_kv_j + \frac{\partial g_2}{\partial I_2}A_{ik}v_k(A_{jl}+A_{lj})v_l + \frac{\partial g_2}{\partial I_3}A_{ik}v_k(A^2_{jl}+A^2_{lj})v_l \\ &+ 2\frac{\partial g_3}{\partial I_1}A^2_{ik}v_kv_j + \frac{\partial g_3}{\partial I_2}A^2_{ik}v_k(A_{jl}+A_{lj})v_l + \frac{\partial g_3}{\partial I_3}A^2_{ik}v_k(A^2_{jl}+A^2_{lj})v_l. \end{aligned}$$

One can easily check that all individual terms on the right-hand side are isotropic function of $(A,\boldsymbol{v})$.

Exercise 4.3.3

Let $B = \boldsymbol{v} \otimes \boldsymbol{v}$, then (4.49) becomes

$$\begin{aligned} &A\boldsymbol{v} \otimes A\boldsymbol{v} + (A^2\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes A^2\boldsymbol{v}) \\ &\quad = (\boldsymbol{v} \cdot A^2\boldsymbol{v} - (\boldsymbol{v} \cdot A\boldsymbol{v}) \operatorname{tr} A - \tfrac{1}{2}(\operatorname{tr} A^2 - (\operatorname{tr} A)^2))\mathit{1} \\ &\quad + (\boldsymbol{v} \cdot A\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{v}) \operatorname{tr} A)A + \tfrac{1}{2}(\operatorname{tr} A^2 - (\operatorname{tr} A)^2)\boldsymbol{v} \otimes \boldsymbol{v} \\ &\quad + \operatorname{tr} A(A\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes A\boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{v})A^2. \end{aligned}$$

The terms on the right-hand side of the equation involve only tensors $(\mathit{1}, A, \boldsymbol{v} \otimes \boldsymbol{v}, (A\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes A\boldsymbol{v}), A^2)$, and the isotropic scalars found in Table 4.1.

Exercise 4.3.4

Let $B = (A\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes A\boldsymbol{v})$, then the left-hand side of (4.49) becomes $2(A^2\boldsymbol{v} \otimes A\boldsymbol{v} + A\boldsymbol{v} \otimes A^2\boldsymbol{v}) + (A^3\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes A^3\boldsymbol{v})$, which by the use of Cayley-Hamilton theorem, the terms involving A^3 can be reduced to combinations of terms of lower polynomial order in A. Similarly, the terms on the right-hand side of the equation (4.49), after elimination of A^3, are all listed in Table 4.1 and Table 4.3, therefore, $(A^2\boldsymbol{v} \otimes A\boldsymbol{v} + A\boldsymbol{v} \otimes A^2\boldsymbol{v})$ can be expressed explicitly in terms of the elements from the tables, and thus need not be included as a generator element.

Likewise, $A^2\boldsymbol{v} \otimes A^2\boldsymbol{v}$ is not needed by the same arguments. In this case, simply take $B = A\boldsymbol{v} \otimes A\boldsymbol{v}$ in (4.49).

Exercise 4.4.1

$$\phi(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{u} \cdot \boldsymbol{v}),$$

$$\boldsymbol{h}(\boldsymbol{u}, \boldsymbol{v}) = h_1\boldsymbol{u} + h_2\boldsymbol{v} + h_3\boldsymbol{u} \times \boldsymbol{v},$$

$$\begin{aligned} S(\boldsymbol{u}, \boldsymbol{v}) &= s_1\mathit{1} + s_2\boldsymbol{u} \otimes \boldsymbol{u} + s_3\boldsymbol{v} \otimes \boldsymbol{v} + s_4(\boldsymbol{u} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{u}) \\ &+ s_5(\boldsymbol{u} \otimes (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{v}) \otimes \boldsymbol{u}) + s_6(\boldsymbol{v} \otimes (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{v}) \otimes \boldsymbol{v}), \end{aligned}$$

where the coefficients are functions of $(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{u} \cdot \boldsymbol{v})$.

Exercise 4.5.1

Since $\boldsymbol{n}$ is a constant unit vector, $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ and $(\boldsymbol{n} \otimes \boldsymbol{n})^2 = \boldsymbol{n} \otimes \boldsymbol{n}$.

1) Regard $\mathcal{Q}$ as an isotropic function of $(\theta, \boldsymbol{g}, \boldsymbol{n})$:

$$\boldsymbol{q} = k_1\boldsymbol{g} + k_2\boldsymbol{n},$$

where k_1, k_2 are functions of $(\theta, \boldsymbol{g} \cdot \boldsymbol{g}, \boldsymbol{g} \cdot \boldsymbol{n})$.

2) Regard $\mathcal{Q}$ as a hemitropic function of $(\theta, \boldsymbol{g}, \boldsymbol{n})$:

$$\boldsymbol{q} = k_1\boldsymbol{g} + k_2\boldsymbol{n} + k_3\boldsymbol{g} \times \boldsymbol{n},$$

where k_1, k_2, k_3 are functions of $(\theta, \boldsymbol{g} \cdot \boldsymbol{g}, \boldsymbol{g} \cdot \boldsymbol{n})$.

3) Regard $\mathcal{Q}$ as an isotropic function of $(\theta, \boldsymbol{g}, \boldsymbol{n} \otimes \boldsymbol{n})$:

$$\boldsymbol{q} = k_1 \boldsymbol{g} + k_2 (\boldsymbol{n} \otimes \boldsymbol{n}) \boldsymbol{g} = k_1 \boldsymbol{g} + k_2 (\boldsymbol{g} \cdot \boldsymbol{n}) \boldsymbol{n},$$

where k_1, k_2 are functions of $(\theta, \boldsymbol{g} \cdot \boldsymbol{g}, (\boldsymbol{g} \cdot \boldsymbol{n})^2)$.

Exercise 4.5.2

By definition, $T = J^{-1} F S_\kappa F^T$, and $J = \sqrt{\det B} = \sqrt{\det C}$, therefore, we obtain, from (4.66),

$$\begin{aligned} T &= t_1 F\boldsymbol{n}_1 \otimes F\boldsymbol{n}_1 + t_2 F\boldsymbol{n}_2 \otimes F\boldsymbol{n}_2 + t_3 F\boldsymbol{n}_3 \otimes F\boldsymbol{n}_3 \\ &\quad + t_4 (F\boldsymbol{n}_1 \otimes BF\boldsymbol{n}_1 + BF\boldsymbol{n}_1 \otimes F\boldsymbol{n}_1) + t_5 (F\boldsymbol{n}_2 \otimes BF\boldsymbol{n}_2 + BF\boldsymbol{n}_2 \otimes F\boldsymbol{n}_2) \\ &\quad + t_6 (F\boldsymbol{n}_3 \otimes BF\boldsymbol{n}_3 + BF\boldsymbol{n}_3 \otimes F\boldsymbol{n}_3) \\ &\quad + t_7 BF\boldsymbol{n}_1 \otimes BF\boldsymbol{n}_1 + t_8 BF\boldsymbol{n}_2 \otimes BF\boldsymbol{n}_2 + t_9 BF\boldsymbol{n}_3 \otimes BF\boldsymbol{n}_3, \end{aligned}$$

where t_i depends on $(\det B, F\boldsymbol{n}_1 \cdot F\boldsymbol{n}_1, F\boldsymbol{n}_2 \cdot F\boldsymbol{n}_2, F\boldsymbol{n}_3 \cdot F\boldsymbol{n}_3,$
$F\boldsymbol{n}_1 \cdot BF\boldsymbol{n}_1, F\boldsymbol{n}_2 \cdot BF\boldsymbol{n}_2, F\boldsymbol{n}_3 \cdot BF\boldsymbol{n}_3)$.

Exercise 5.3.1

From (5.25), $\theta d\eta = d\varepsilon - \dfrac{p}{\rho^2} d\rho$, hence

$$d(\theta\eta) = \theta d\eta + \eta d\theta = d\varepsilon - \frac{p}{\rho^2} d\rho + \eta d\theta, \text{ and}$$

$$\theta d(\rho\eta) = \rho\theta d\eta + \theta\eta d\rho = \rho d\varepsilon - \frac{p}{\rho} d\rho + \theta\eta d\rho = d(\rho\varepsilon) - (\varepsilon - \theta\eta + \frac{p}{\rho}) d\rho.$$

Exercise 5.3.2

Note that for any scalar function $\hat{f}(B) = \hat{f}(FF^T) = f(F)$, we have

$$\frac{\partial f}{\partial F}[S] = \frac{\partial \hat{f}}{\partial B}[SF^T + FS^T] = 2 \frac{\partial \hat{f}}{\partial B} F[S],$$

since for any $S, A \in \mathcal{L}(V)$, $A[S] = A \cdot S = \operatorname{tr} AS^T = A^T \cdot S^T$. Hence, we obtain $2 \dfrac{\partial \hat{f}}{\partial B} B = \dfrac{\partial f}{\partial F} F^T$, which leads to the relations (5.32) easily.

To show (5.34), use the formulae (A.54).

Exercise 5.4.1

The material is incompressible and inextensible in a certain direction. The constraint can be specified by two conditions,

$$\tilde{\mu}_1(C) = \det C - 1 = 0, \quad \tilde{\mu}_2(C) = \boldsymbol{e} \cdot C\boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{e} = 0,$$

from which we have $\dfrac{\partial \tilde{\mu}_1}{\partial C} = (\det C) C^{-T}$, $\dfrac{\partial \tilde{\mu}_2}{\partial C} = \boldsymbol{e} \otimes \boldsymbol{e}$. Hence, by (5.56),

$$N = -p\mathit{1} + \lambda F\boldsymbol{e} \otimes F\boldsymbol{e}.$$

Exercise 5.4.2

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a basis of V, then we can write $\mu(C) = C - 1 = 0$ in component form as $\mu_{ij}(C) = \boldsymbol{e}_i \cdot C\boldsymbol{e}_j - \boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$. We have $\mu_{ij}(C) = \mu_{ji}(C)$, and the reaction stress is given by

$$N = \sum_{i,j} \lambda^{ij} F \frac{\partial \mu_{ij}}{\partial C} F^T = \sum_{i,j} \lambda^{ij} F(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) F^T = \sum_{i,j} \lambda^{ij} F\boldsymbol{e}_i \otimes F\boldsymbol{e}_j = \Lambda.$$

Since F is non-singular, $\{F\boldsymbol{e}_1, F\boldsymbol{e}_2, F\boldsymbol{e}_3\}$ is also a basis of V, and $\lambda^{ij} = \lambda^{ji}$ are arbitrary parameters, therefore, Λ is an arbitrary symmetric tensor in $\mathcal{L}(V)$.

Exercise 5.5.1

We can write $u = u(v(u, w), w) = \tilde{u}(u, w)$.

1) $1 = \dfrac{\partial \tilde{u}}{\partial u} = \dfrac{\partial u}{\partial v}\Big|_w \dfrac{\partial v}{\partial u}\Big|_w .$

2) $0 = \dfrac{\partial \tilde{u}}{\partial w} = \dfrac{\partial u}{\partial v}\Big|_w \dfrac{\partial v}{\partial w}\Big|_u + \dfrac{\partial u}{\partial w}\Big|_v$, which gives $\dfrac{\partial u}{\partial v}\Big|_w \dfrac{\partial v}{\partial w}\Big|_u \dfrac{\partial w}{\partial u}\Big|_v = -1.$

Exercise 5.5.2

1) It follows from $(5.30)_1$ with $dv = -(1/\rho^2)d\rho$.

2) We have, from (5.72), $\dfrac{\partial p}{\partial \rho}\Big|_\theta = -\dfrac{1}{\rho^2}\dfrac{\partial p}{\partial v}\Big|_\theta \geq 0$, and from (1), $-p = \dfrac{\partial \psi}{\partial v}\Big|_\theta$. Therefore,

$$-\frac{\partial p}{\partial v}\Big|_\theta = \frac{\partial^2 \psi}{\partial v^2}\Big|_\theta \geq 0.$$

Exercise 5.5.3

From (5.25), we have $\theta d\eta = d\varepsilon + p\,dv$, therefore

$$\begin{aligned}
\theta d\eta &= \frac{\partial \varepsilon}{\partial \theta}\Big|_v d\theta + \Big(\frac{\partial \varepsilon}{\partial v}\Big|_\theta + p\Big) dv \\
&= \frac{\partial \varepsilon}{\partial \theta}\Big|_v d\theta + \Big(\frac{\partial \varepsilon}{\partial v}\Big|_\theta + p\Big)\Big(\frac{\partial v}{\partial \theta}\Big|_p d\theta + \frac{\partial v}{\partial p}\Big|_\theta dp\Big) \\
&\overset{(5.26)}{=} \Big(\frac{\partial \varepsilon}{\partial \theta}\Big|_v + \theta \frac{\partial p}{\partial \theta}\Big|_v \frac{\partial v}{\partial \theta}\Big|_p\Big) d\theta + \theta \frac{\partial p}{\partial \theta}\Big|_v \frac{\partial v}{\partial p}\Big|_\theta dp
\end{aligned}$$

Therefore, we obtain

$$\theta \frac{\partial \eta}{\partial \theta}\Big|_v = \frac{\partial \varepsilon}{\partial \theta}\Big|_v = c_v \quad \text{and} \quad \theta \frac{\partial \eta}{\partial \theta}\Big|_p = \frac{\partial \varepsilon}{\partial \theta}\Big|_v + \theta \frac{\partial p}{\partial \theta}\Big|_v \frac{\partial v}{\partial \theta}\Big|_p .$$

By definition of c_p and the use of identity (2) of Exercise 5.5.1, it follows that

$$c_p = c_v - \theta\, \frac{\partial p}{\partial v}\bigg|_\theta \Big(\frac{\partial v}{\partial \theta}\bigg|_p\Big)^2, \text{ and since } \frac{\partial p}{\partial v}\bigg|_\theta \leq 0, \text{ we have } c_p \geq c_v.$$

Furthermore, by definition of α and κ_T, we can write the relation as $c_p = c_v + \theta v \alpha^2/\kappa_T$, and since $c_v \geq 0$, it follows that

$$\alpha^2 \leq \frac{1}{\theta v} c_p \kappa_T.$$

Exercise 5.5.4

The boundary conditions are given by

$$\theta = \theta_o, \quad T\boldsymbol{n} = -p_o \boldsymbol{n} \quad \text{on} \quad \partial\mathcal{V}.$$

The energy balance becomes

$$\frac{d}{dt}\int_{\mathcal{V}} \rho(\varepsilon + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\, dv + \int_{\partial\mathcal{V}} \boldsymbol{q}\cdot\boldsymbol{n}\, da + p_o \int_{\partial\mathcal{V}} \boldsymbol{v}\cdot\boldsymbol{n}\, da = 0.$$

Elimination of the heat flux by the use of the entropy inequality, with $\boldsymbol{\Phi} = \boldsymbol{q}/\theta_o$ at the boundary, and the relation (2.10), gives

$$\frac{d}{dt}\int_{\mathcal{V}} \rho(\varepsilon - \theta_o \eta + \frac{p_o}{\rho} + \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\, dv \leq 0.$$

Exercise 6.5.1

We denote the unit normal and the unit tangent of the slanted surface, as indicated in Fig. 6.5, by

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\kappa^2}}(1, -\kappa, 0), \quad \boldsymbol{m} = \frac{1}{\sqrt{1+\kappa^2}}(\kappa, 1, 0).$$

Then the normal and tangential stresses are given by

$$N = \boldsymbol{n}\cdot T\boldsymbol{n}\ , \text{ and } \ S = \boldsymbol{m}\cdot T\boldsymbol{n}.$$

The solution follows from direct calculations and the use of (6.41).

Exercise 6.5.2

Reference: see [71] Sec. 57, and [76] p. 307.

Exercise 6.6.1

Relative to the basis $\{\hat{\boldsymbol{e}}_i\}$, we have

$$[\widehat{T}^{ij}] = \begin{bmatrix} 0 & \widehat{T}^{12} & 0 \\ \widehat{T}^{21} & 0 & 0 \\ 0 & 0 & \widehat{T}^{33} \end{bmatrix}, \quad [\widehat{B}^{ij}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix},$$

$$[\,\hat{g}_{ij}\,] = \begin{bmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compare the expressions on both sides of $[T^{ij}][g_{jk}][B^{kl}] = [B^{ij}][g_{jk}][T^{kl}]$, by direct matrix multiplications, gives the relation (6.10).

Exercise 6.6.2

From (6.43), we obtain

$$[C_{\langle i\alpha\rangle}] = \begin{bmatrix} \lambda_1^2 & \kappa\lambda_1\lambda_2 & 0 \\ \kappa\lambda_1\lambda_2 & (1+\kappa^2)\lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & \kappa\lambda_1\lambda_2 & 0 \\ \kappa\lambda_1\lambda_2 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix},$$

which admits the eigenvectors and the corresponding eigenvalues,

$$\boldsymbol{n}_1 = \frac{1}{\sqrt{2}}(1,1,0), \quad \boldsymbol{n}_2 = \frac{1}{\sqrt{2}}(1,-1,0), \quad \boldsymbol{n}_3 = (0,0,1),$$

$$\alpha_1 = \lambda_1^2 + \kappa\lambda_1\lambda_2, \quad \alpha_2 = \lambda_1^2 - \kappa\lambda_1\lambda_2, \qquad \alpha_3 = \lambda_3^2.$$

From Fig. 6.6, we have

$$\lambda_2 = \lambda_1 \cos 2\theta, \quad \kappa\lambda_2 = \lambda_1 \sin 2\theta,$$

and hence,

$$\alpha_1 = \lambda_1^2(1+\sin 2\theta) = \lambda_1^2(\cos\theta + \sin\theta)^2,$$
$$\alpha_2 = \lambda_1^2(1-\sin 2\theta) = \lambda_1^2(\cos\theta - \sin\theta)^2.$$

Since $\theta < \pi/4$, $\cos\theta > \sin\theta$, therefore, we can easily obtain the expression of U by

$$\begin{aligned} U &= \sqrt{\alpha_1}\, \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \sqrt{\alpha_2}\, \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \sqrt{\alpha_3}\, \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 \\ &= \lambda_1(\cos\theta + \sin\theta)\, \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \lambda_1(\cos\theta - \sin\theta)\, \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \lambda_3\, \boldsymbol{n}_3 \otimes \boldsymbol{n}_3. \end{aligned}$$

Exercise 6.7.1

Reference: see [71] Sec. 57, and [76] p. 310.

Exercise 6.8.1

Reference: see [42].

Exercise 7.3.1

$$[X] = [\,\dot\rho \quad \dot v_n \quad \dot\theta \quad (\theta_{,n})^{\cdot} \quad \dot D_{mn} \quad \rho_{,n} \quad \theta_{,mn} \quad D_{mn,k}\,]^T,$$

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho\delta_{in} & 0 & 0 & 0 & -\dfrac{\partial T_{in}}{\partial\rho} & -\dfrac{\partial T_{im}}{\partial\theta_{,n}} & -\dfrac{\partial T_{ik}}{\partial D_{mn}} \\ \rho\dfrac{\partial\varepsilon}{\partial\rho} & 0 & \rho\dfrac{\partial\varepsilon}{\partial\theta} & \rho\dfrac{\partial\varepsilon}{\partial\theta_{,n}} & \rho\dfrac{\partial\varepsilon}{\partial D_{mn}} & \dfrac{\partial q_n}{\partial\rho} & \dfrac{\partial q_m}{\partial\theta_{,n}} & \dfrac{\partial q_k}{\partial D_{mn}} \end{bmatrix},$$

$$[B] = \left[\rho D_{kk} \quad -\frac{\partial T_{ik}}{\partial\theta}\theta_{,k} \quad \frac{\partial q_k}{\partial\theta}\theta_{,k} - T_{ik}D_{ik}\right]^T,$$

$$[\alpha] = \left[\rho\frac{\partial \eta}{\partial \rho} \quad 0 \quad \rho\frac{\partial \eta}{\partial \theta} \quad \rho\frac{\partial \eta}{\partial \theta_{,n}} \quad \rho\frac{\partial \eta}{\partial D_{mn}} \quad \frac{\partial \Phi_n}{\partial \rho} \quad \frac{\partial \Phi_m}{\partial \theta_{,n}} \quad \frac{\partial \Phi_k}{\partial D_{mn}} \right],$$

$$[\beta] = \left[\frac{\partial \Phi_k}{\partial \theta}\theta_k \right].$$

Exercise 7.3.2

$$\begin{aligned}
f_\rho &= -\rho v_{k,k}, \\
f_{v_i} &= -v_k v_{i,k} + \frac{1}{\rho}\Big(\frac{\partial T_{ik}}{\partial \rho}\rho_{,k} + \frac{\partial T_{ik}}{\partial \theta}\theta_{,k} + \frac{\partial T_{ik}}{\partial \theta_{,n}}\theta_{,nk} + \frac{\partial T_{ik}}{\partial D_{mn}}D_{mn,k} \Big), \\
f_\varepsilon &= -v_k\Big(\frac{\partial \varepsilon}{\partial \rho}\rho_{,k} + \frac{\partial \varepsilon}{\partial \theta}\theta_{,k} + \frac{\partial \varepsilon}{\partial \theta_{,n}}\theta_{,nk} + \frac{\partial \varepsilon}{\partial D_{mn}}D_{mn,k} \Big) \\
&\quad - \frac{1}{\rho}\Big(\frac{\partial q_k}{\partial \rho}\rho_{,k} + \frac{\partial q_k}{\partial \theta}\theta_{,k} + \frac{\partial q_k}{\partial \theta_{,n}}\theta_{,nk} + \frac{\partial q_k}{\partial D_{mn}}D_{mn,k} - T_{ik}v_{i,k} \Big).
\end{aligned}$$

Exercise 8.2.1

We shall show that $\boldsymbol{\Lambda}\cdot\dfrac{\partial \boldsymbol{u}}{\partial t} = \dfrac{\partial \rho\eta}{\partial t}$.

$$\begin{aligned}
\boldsymbol{\Lambda}\cdot\frac{\partial \boldsymbol{u}}{\partial t} &= -\frac{1}{\theta}\Big(\varepsilon - \theta\eta + \frac{p}{\rho} - \frac{v^2}{2}\Big)\frac{\partial \rho}{\partial t} - \frac{1}{\theta}v_i\frac{\partial \rho v_i}{\partial t} + \frac{1}{\theta}\frac{\partial}{\partial t}\Big(\rho\varepsilon + \frac{1}{2}\rho v^2\Big) \\
&= \eta\frac{\partial \rho}{\partial t} + \frac{\rho}{\theta}\Big(\frac{\partial \varepsilon}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t}\Big) \overset{(8.14)}{=} \eta\frac{\partial \rho}{\partial t} + \rho\frac{\partial \eta}{\partial t} = \frac{\partial \rho\eta}{\partial t}.
\end{aligned}$$

The proof for $\boldsymbol{\Lambda}\cdot\dfrac{\partial \boldsymbol{H}_j}{\partial t} = \dfrac{\partial}{\partial t}(\rho\eta v_j)$ is similar.

Exercise 8.2.2

From (8.15) and (8.16), we have

$$\begin{aligned}
\delta\boldsymbol{u}\cdot\delta\boldsymbol{\Lambda} &= -(\delta\rho)\,\delta\Big(\frac{\varepsilon}{\theta} - \eta + \frac{p}{\rho\theta} - \frac{v^2}{2\theta}\Big) - \delta(\rho v_i)\,\delta\Big(\frac{v_i}{\theta}\Big) + \delta\Big(\rho\varepsilon + \frac{\rho v^2}{2}\Big)\,\delta\Big(\frac{1}{\theta}\Big) \\
&= -\Big(\frac{1}{\theta}\frac{\partial \varepsilon}{\partial \rho} - \frac{\partial \eta}{\partial \rho} - \frac{p}{\theta\rho^2} + \frac{1}{\rho\theta}\frac{\partial p}{\partial \rho}\Big)(\delta\rho)^2 - \frac{\rho}{\theta}(\delta v_i)(\delta v_i) \\
&\quad - \frac{\rho}{\theta^2}\frac{\partial \varepsilon}{\partial \theta}(\delta\theta)^2 - \Big(\frac{1}{\theta}\frac{\partial \varepsilon}{\partial \theta} - \frac{\partial \eta}{\partial \theta} + \frac{\rho}{\theta^2}\frac{\partial \varepsilon}{\partial \rho} - \frac{1}{\theta^2}\frac{p}{\rho} + \frac{1}{\rho\theta}\frac{\partial p}{\partial \theta}\Big)(\delta\rho)(\delta\theta) \\
&= -\frac{\rho}{\theta}\Big\{(\delta v_k)(\delta v_k) + \frac{1}{\theta}\frac{\partial \varepsilon}{\partial \theta}(\delta\theta)^2 + \frac{1}{\rho^2}\frac{\partial p}{\partial \rho}(\delta\rho)^2\Big\},
\end{aligned}$$

where in the last passage the Gibbs relation (5.25) and its integrability condition (5.26) are used.

Exercise 8.2.3

From (8.26), we obtain

$X(\boldsymbol{u}+\boldsymbol{v}) = X(\boldsymbol{v}+\boldsymbol{u}) = X(\boldsymbol{u})X(\boldsymbol{v}) = X(\boldsymbol{v})X(\boldsymbol{u}),$

and

$X(\boldsymbol{v}-\boldsymbol{v}) = X(\boldsymbol{v})X(-\boldsymbol{v}) = X(0) = \mathit{1}.$

Exercise 8.3.1

Reference: see [64].

Exercise 8.5.1

Let $\boldsymbol{\Lambda} = (\hat{\Lambda}_0, \hat{\Lambda}_i, \hat{\Lambda}_{ij}, \hat{\Lambda}_{llj})$ and $\boldsymbol{u} = (F_0, F_i, F_{ij}, F_{iij})$, then, from (8.38), we can write $(8.4)_1$ as

$$\begin{aligned} dh &= \hat{\Lambda}_0\, d\rho + \hat{\Lambda}_i\, d(\rho v_i) + \hat{\Lambda}_{ij}\, d(\rho_{ij} + \rho v_i v_j) + \hat{\Lambda}_{llj}\, d(\rho_{iij} + 3\rho_{(ii} v_{j)} + \rho v_i v_i v_j) \\ &= \Lambda\, d\rho + \Lambda_{ij}\, d\rho_{ij} + \lambda_j\, d\rho_{iij} + (\rho \Lambda_j + 3\lambda_{(i}\rho_{ij)})\, dv_j, \end{aligned}$$

where we have defined the internal Lagrange multipliers as

$\Lambda = \hat{\Lambda}_0 + \hat{\Lambda}_i v_i + \hat{\Lambda}_{ij} v_i v_j + \hat{\Lambda}_{llj} v_i v_i v_j,$

$\Lambda_i = \hat{\Lambda}_i + 2\hat{\Lambda}_{ij} v_j + 3\hat{\Lambda}_{ll(i} v_j v_{j)},$

$\Lambda_{ij} = \hat{\Lambda}_{ij} + +3\hat{\Lambda}_{llk} v_{(k}\delta_{ij)},$

$\lambda_j = \hat{\Lambda}_{llj}.$

Since $h = \rho\eta$ is velocity-independent, so do the internal Lagrange multipliers defined above. Moreover, we obtain (8.68),

$\rho \Lambda_j + 3\lambda_{(i}\rho_{ij)} = 0.$

The proof of (8.69) is similar (see [46] Sec. 6).

Exercise 8.5.2

See Exercise 8.2.2.