Dynamics of a virtual virus infection process involving a spatial distribution of interacting computers

R. CIPOLATTI* and J. LÓPEZ GONDAR†

†Departamento de Métodos Matemáticos
‡Departamento de Matemática Aplicada, Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530, CEP 21945-970, Rio de Janeiro, Brazil

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In this work we introduce a theoretical model for describing the dynamics of a virtual virus infection process in a set of interacting computers, considering spatial densities of susceptible and infected equipments distributed in a two dimensional region. We consider here two possible propagation mechanisms: the transmission and reception of electronic mails and also the exchange of softwares by means of recording devices as compact disks or the commonly used floppy disks. The model gives a good idea of the infection process and trends reproducing, satisfactorily, some previously obtained results concerning the homogeneous mixing case. The new nonlinear integral equation we obtain here presents certain interesting features which are explored and enlightened in this article.

Keywords: Virtual viruses; Nonlinear delay integral equations; Population equation

AMS Subject Classifications: 45G10; 49K25; 92D99

1. Introduction

The infection of computers by virtual viruses is a present day problem. Although a lot of successful effort has been devoted to the development of virtual vaccines during the last years, progress in understanding the propagation process itself, quantitatively, has been rather slow. Recently [4], a mathematical model for a virus infection propagation process in a system of interacting computers has been reported by the authors, but only the simplest homogeneous mixing case was considered there. Although in relatively small communities some form of homogeneous mixing can be assumed, such an
approach has no sense when dealing with widely dispersed populations. For this reason, in the present article we should consider a geographical distribution for the interacting computers in order to obtain a more realistic description of the aforementioned process.

The virus propagation mechanisms that we consider here are: (1) the transmission and reception of infected messages using the international electronic network and (2) the exchange of infected softwares by means of recording devices like compact disks or floppy disks. In our simplified model, which ignores finer details, the recording device (when it is the case) is considered as a simple carrier, not permanently infected, which transmits the virus only once (if it is so); i.e., after the first transmission of a certain program (infected or not), the contents in the recording device are deleted before it could be used again. This fact introduces significant differences regarding those cases of biological diseases transmitted by vectors and requires a reformulation of the usual contact-propagation equations (see [3]).

In order to describe the space–time evolution of the virtual infection process, we propose a generalization of the model developed by us in [4], which consists of an integral equation involving densities of susceptible and infected computers.

This article is organized as follows: in section 2 we introduce a set of hypotheses which will support the modeling process, define those functions related to the magnitudes involved and construct the theoretical model. Section 3 is devoted to perform a mathematical analysis of the integral equation obtained: existence, uniqueness and asymptotic behavior of the solutions. In section 4 we consider the problem of existence of nontrivial stationary solutions. Finally, section 5 contains numerical experiments (including the interesting case of regional software poles), discussion and our conclusions.

2. The mathematical model

Let us consider a two-dimensional bounded region $\Omega$ containing $N$ computers spatially distributed over its surface. The closed host population of total size $N$ will be used as a normalization factor and, being so, we introduce a surface density of computers $\rho(t, x)$, where $x = (x_1, x_2) \in \Omega$ and $\int_{\Omega} \rho(t, x) \, dx = 1$, to describe the corresponding spatial distribution. In the present approach we shall assume to deal with a new kind of virus infection for which there is no vaccine available. Being so, the unique way to restore the functions of an infected computer is by deleting its hard disk (with which the computer becomes vulnerable again). It should be pointed out that removal of infected computers or existence of latent periods won’t be considered here.

The whole population of computers is divided into two classes:

(I) the infective class. This includes all infected individuals because, in the absence of a latent period, any infected individual immediately becomes an infective one, displaying a potential capacity of transmitting the disease to other susceptible individuals;

(S) the susceptible class; i.e., those individuals capable of contracting the disease, becoming themselves infected.
The probability for an infected (resp. susceptible) computer to exist at time \( t \) in the host population will be denoted by \( i(t) \) (resp. \( s(t) \)); so we have \( i(t) + s(t) = 1 \). Because the interacting computers are not necessarily homogeneously mixed, one should introduce surface densities of probability \( I(t, x) \) and \( S(t, x) \) such that

\[
i(t) = \int_{\Omega} I(t, x) \, dx \quad \text{and} \quad s(t) = \int_{\Omega} S(t, x) \, dx,
\]

where \( I(t, x) + S(t, x) = \rho(t, x) \).

We suppose that, when using recording devices, each one of them enters only twice in the process within a time interval \( \eta \) which depends on the geographical position of the two computers involved in the exchange of programs. Once the program is transferred from one computer to another, we assume that the contents in the recording device are deleted and, only then, it is incorporated again to the original recording chain. In the case of electronic mails, the parameter \( \eta \) should represent the mean time elapsed between the emission and the effective reception of the message. This process allows a computer located at \( y \in \Omega \) to interact with another one located at \( x \in \Omega \) after a time delay \( \eta(x, y) \). Such an interaction is characterized by a certain weight coefficient \( p(x, y) \geq 0 \). If, for example, we want to describe the probability density for a susceptible computer at \( (t, x) \) to receive a virtual contact deriving from a set of infected computers located at different points of the whole territory, we write

\[
Q_{I \to S}(t, x) = S(t, x) \int_{\Omega} p(x, y) I(t - \eta(x, y), y) \, dy,
\]

while the probability of infection itself is obtained by integrating the above expression in \( x \) over \( \Omega \).

A table containing all the possible “contact” events and its corresponding surface densities of probability is shown below.

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability surface density</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I \to I )</td>
<td>( I(t, x) \int_{\Omega} p(x, y) I(t - \eta(x, y), y) , dy )</td>
</tr>
<tr>
<td>( S \to S )</td>
<td>( S(t, x) \int_{\Omega} p(x, y) S(t - \eta(x, y), y) , dy )</td>
</tr>
<tr>
<td>( I \to S )</td>
<td>( S(t, x) \int_{\Omega} p(x, y) I(t - \eta(x, y), y) , dy )</td>
</tr>
<tr>
<td>( S \to I )</td>
<td>( I(t, x) \int_{\Omega} p(x, y) S(t - \eta(x, y), y) , dy )</td>
</tr>
</tbody>
</table>

The sum of the above probability surface densities, integrated over \( \Omega \), should give just 1. It is not difficult to show that such a condition leads to

\[
\int_{\Omega} \left( \int_{\Omega} \rho(t, x)p(x, y)\rho(t - \eta(x, y), y) \, dy \right) \, dx = 1.
\]

Equation (2) must hold for a suitable choice of the non-negative weight function \( p(x, y) \).

Among the aforementioned processes we are only interested in the third one (equation (1)), because it reflects the probability of contamination, which is our objective.
We consider that each computer has a certain average e-mail receiving-frequency associated with it, as well as a characteristic recording frequency concerning softwares exchange. Denoting by \( N_r(t, x) \) such a frequency (depending on the case under consideration), we may write that, after a time interval \( \Delta t \) small enough, the probability surface density of infected computers which is increased to the one already existing at \((t, x)\) is given, approximately, as

\[
I(t, x) = \int_\Omega p(x, y) I(t - \eta(x, y), y) \, dy,
\]

where

\[
\alpha(t, x) = \frac{N_r(t, x)}{N}.
\]

Dividing by \( \Delta t \), we obtain the surface rate of the infection process at time \( t \), which could be written as

\[
g(t, x) = \alpha(t, x) \left[ \rho(t, x) - I(t, x) \right] \int_\Omega p(x, y) I(t - \eta(x, y), y) \, dy.
\]

Following references [3,4], we now introduce a function of two variables \( P(t, \tau) \), \( P: \Lambda \to [0, 1] \), where

\[
\Lambda = \{(t, \tau) \in \mathbb{R}^2; \tau \geq t \}.
\]

It represents the probability of a computer infected at \( \tau \) to remain infective at \( t \). Such a function, of course, is related to the time elapsed between the infection of a computer and its recovering after erasing the hard disk. Considering a maximum recovering time \( T \), common for all \( x \in \Omega \), the function \( P(t, \tau) \) is supposed to have the following general properties:

\[
\begin{align*}
(\text{i}) & \quad P(t, \tau) = 1 \quad \text{if} \quad \tau = t; \\
(\text{ii}) & \quad P(t, \tau) = 0 \quad \text{if} \quad \tau \leq t - T; \\
(\text{iii}) & \quad P(t, \tau) \quad \text{is a monotone decreasing function of} \ t \ \text{for each fixed} \ \tau.
\end{align*}
\]

From now on, the reasoning is the same followed in [4] to get the general evolution of infection processes in the framework of SIS models, namely

\[
I(t, x) = \int_{t-T}^{t} g(\tau, x) P(t, \tau) \, d\tau.
\]

In our case here, the probability density for a computer to be infective at \((t, x)\) is given by

\[
I(t, x) = \int_{t-T}^{t} \alpha(\tau, x) \left( \rho(\tau, x) - I(\tau, x) \right) G[I(\tau, x)] P(t, \tau) \, d\tau, \quad t > 0,
\]
where $G[I]$ is the nonlocal operator defined by

$$G[I](\tau, x) = \int_{\Omega} p(x, y) I(\tau - \eta(x, y), y) dy.$$  \hspace{1cm} (9)

Equation (8) is a new kind of delay integral equation and its solution for $t \geq 0$ depends on data defined in the past. Notice that it reduces to the simplest case of homogeneously mixed computers if $p(x, y) = 1$, $\rho(t, x) = 1$, $\alpha$ is independent of $x$, and $\eta(x, y) = t_0$. This particular case is just the one already studied in [4].

3. Existence, uniqueness and asymptotic behavior

In this section we consider the problems of existence, uniqueness, and asymptotic behavior of solutions for equation (8). We begin by introducing the notation and the functional framework. Although for applications it would be natural to consider a more restrictive setting, we proceed with our analysis in a general background, where the restrictions for applications are merely particular cases.

Let $\Omega$ be a connected subset (not necessarily bounded) of $\mathbb{R}^N$, $(N \geq 1)$. For $\eta^* > 0$ given, we consider the space

$$\mathcal{E} = C_b([-T - \eta^*, + \infty); L^1(\Omega)),$$

which is a Banach space for the norm

$$||f||_{\mathcal{E}} = \max\left\{\int_{\Omega} |f(\tau, y)| \, dy ; \, \tau \geq -T - \eta^*\right\}.$$  \hspace{1cm} (10)

Let $Q = [-T - \eta^*, + \infty) \times \Omega$. In addition to (7), we assume that:

(i) $\alpha \in L^\infty(Q)$ is such that $\alpha(t, x) \geq 0$;
(ii) $p \in L^\infty(\Omega \times \Omega)$ is such that $p(x, y) \geq 0$;
(iii) $\eta \in L^\infty(\Omega \times \Omega)$ is such that $0 \leq \eta(x, y) \leq \eta^* < +\infty$; \hspace{1cm} (11)
(iv) $\rho \in \mathcal{E}$ is such that $\rho \geq 0$, $\int_{\Omega} \rho(\tau, y) \, dy = 1$ $\tau \geq -T - \eta^*$.

We notice that, although there are no particular restrictions on the upper bound of $p$, we may assume that

$$0 \leq p(x, y) \leq 1 \text{ a.e. in } \Omega \times \Omega.$$  \hspace{1cm} (12)

Indeed, if $p^* = \text{ess sup}\{p(x, y) ; \, x, y \in \Omega\}$, the equation (8) may be written as

$$I(t, x) = \int_{t-T}^{t} \tilde{\alpha}(\tau, x) \left( \rho(\tau, x) - I(\tau, x) \right) G[I](\tau, x) P(t, \tau) \, d\tau, \quad t > 0,$$
where $\tilde{\alpha}(\tau, x) = \alpha(\tau, x)p^*$ and $\tilde{G}[I]$ is defined as

$$\tilde{G}[I](\tau, x) = \frac{1}{p^*} \int_\Omega p(x, y) I(\tau - \eta(x, y), y) \, dy.$$ 

**Definition 3.1** We will say that a function $I: [-T - \eta^*, +\infty) \times \Omega \to \mathbb{R}$ is a solution generated by $I_0 \in \mathcal{E}$ if:

$$
\begin{align*}
I(\tau, x) &= I_0(\tau, x) \text{ in } [-T - \eta^*, 0] \times \Omega; \\
I &\text{ satisfies equation (8) for } t > 0.
\end{align*}
$$

Let $\eta_* = \inf\{\eta(x, y); x, y \in \Omega\}$. The next theorem concerns the existence and uniqueness of solution of equation (8) in the case $\eta_* > 0$.

**Theorem 3.2** We assume that $\eta_* > 0$ and $I_0 \in \mathcal{E}$. Under the conditions (7), (11) and (12), (8) has a unique solution $\bar{I} \in C([T - \eta^*, +\infty); L^1(\Omega))$ generated by $I_0$.

**Proof** If $I_1, I_2 \in C([-T - \eta^*, +\infty); L^1(\Omega))$ are two solutions generated by $I_0$, then we have for $0 < t \leq \eta_*$

$$I_1(t, x) - I_2(t, x) = \int_0^t \alpha(\tau, x)(I_2(\tau, x) - I_1(\tau, x)) \tilde{G}[I_0](\tau, x)P(t, \tau) \, d\tau.$$ 

Let $\phi$ be the function defined by $\phi(t) = \int_\Omega I_1(t, y) - I_2(t, y) \, dy$. Since $0 \leq P(t, \tau) \leq 1$, it follows from (12) and Fubini’s theorem that

$$\phi(t) \leq \alpha^*||I_0||e \int_0^t \phi(\tau) \, d\tau, \quad 0 < t \leq \eta_*,$$

where $\alpha^* = ||\alpha||_{L^\infty(\Omega)}$. Since the function $t \mapsto \int_0^t \phi(\tau) \, d\tau$ is positive and differentiable, we conclude easily that $\phi(t) = 0$ on $[0, \eta_*]$. The same arguments can be repeated for $(k - 1)\eta_* < t \leq k\eta_*$, for all $k \in \mathbb{N}$, and the proof of uniqueness is achieved.

We divide the proof of existence into three steps.

**Step 1 – Introducing the notation:** Let $a, b$ be two real numbers such that $-T - \eta^* < a < b$ and consider the set

$$\mathcal{E}_{a, b}(I_0) = \left\{ I \in C([-T - \eta^*, b]; L^1(\Omega)) \mid I = I_0 \text{ on } [-T - \eta^*, a] \times \Omega \right\}.$$
It is easily seen that \( \mathcal{E}_{a,b}(I_0) \) is a nonempty closed subset of the Banach space \( C([-T - \eta^*, b]; L^1(\Omega)) \).

We introduce the operator \( \Phi: \mathcal{E}_{a,b}(I_0) \to \mathcal{E}_{a,b}(I_0) \) defined by \( \Phi[I](t, x) = I_0(t, x) \) a.e. in \([-T - \eta^*, a] \times \Omega \) and

\[
\Phi[I](t, x) = \int_{t-T}^t \alpha(\tau, x)(\rho(\tau, x) - I(\tau, x))G[\tau, x]P(t, \tau)\,d\tau
\]  

(15)
a.e. in \((a, b) \times \Omega\).

**Step 2 – Getting a fixed point:** Let \( I, \tilde{I} \in \mathcal{E}_{0,b}(I_0) \) and consider the operator \( \Phi \) defined in (15) with \( a = 0 \) and

\[
b = \min\{T, \eta^*\}.
\]

(16)

Then, \( \Phi[I](t, x) - \Phi[\tilde{I}](t, x) = 0 \) a.e. in \([-T - \eta^*, 0] \times \Omega \) and

\[
\Phi[I](t, x) - \Phi[\tilde{I}](t, x) = \int_0^t \alpha(\tau, x)(\tilde{I}(\tau, x) - I(\tau, x))G[I_0](\tau, x)P(t, \tau)\,d\tau,
\]

(17)
a.e. in \((0, b) \times \Omega\). From (12) and \( 0 \leq P(t, \tau) \leq 1 \) for all \((t, \tau) \in \Lambda\), it follows that

\[
|\Phi[I](t, x) - \Phi[\tilde{I}](t, x)| \leq \alpha^*||I_0||_\mathcal{E} \int_0^t |I(\tau, x) - \tilde{I}(\tau, x)|\,d\tau
\]
a.e. in \((0, b) \times \Omega\). Integrating over \( \Omega \) we have

\[
||\Phi[I](t, \cdot) - \Phi[\tilde{I}](t, \cdot)||_{L^1(\Omega)} \leq \alpha^*||I_0||_\mathcal{E} \int_0^t ||I(\tau, \cdot) - \tilde{I}(\tau, \cdot)||_{L^1(\Omega)}\,d\tau
\]

\[
\leq t\alpha^*||I_0||_\mathcal{E}||\tilde{I} - I||_\mathcal{E}
\]

(18)

A recurrent argument using (17) and (18) gives, for every \( t \in (0, b)\),

\[
|\Phi^k[I](t, x) - \Phi^k[\tilde{I}](t, x)| \leq \frac{1}{k!}(t\alpha^*||I_0||_\mathcal{E})^k||\tilde{I} - I||_\mathcal{E},
\]

(19)

from which we obtain easily

\[
||\Phi^k[I] - \Phi^k[\tilde{I}]||_\mathcal{E} \leq \frac{1}{k!}(b\alpha^*||I_0||_\mathcal{E})^k||\tilde{I} - I||_\mathcal{E}.
\]

(20)

Choosing \( k \) large enough, it follows from (20) that \( \Phi^k \) is a contraction in \( \mathcal{E}_{0,b}(I_0) \) and the Banach fixed point theorem assures the existence of a unique fixed point \( I_1 \) for \( \Phi^k \) in \( \mathcal{E}_{0,b}(I_0) \). More precisely, there exists a unique \( I_1 \in \mathcal{E}_{0,b}(I_0) \) such that \( \Phi^k[I_1] = I_1 \). Since \( \Phi^{k+1}[I_1] = \Phi^k[\Phi[I_1]] = \Phi[I_1] \), it follows from the uniqueness that \( \Phi[I_1] = I_1 \) in \( \mathcal{E}_{0,b}(I_0) \).
Step 3 – Extending the solution: We consider now \( \Phi : \mathcal{E}_{b, 2b}(I_1) \to \mathcal{E}_{b, 2b}(I_1) \), where \( b \) is defined in (16) and \( I_1 \in \mathcal{E}_{0, b}(I_0) \) is the fixed point obtained. Then, the same arguments of the previous step hold and we obtain a fixed point \( I_2 \in \mathcal{E}_{b, 2b}(I_1) \) of \( \Phi \) such that \( I_2(t, x) = I_1(t, x) \) a.e. \((t, x) \in [-T - \eta^*, b] \times \Omega \). 

Arguing by induction, we obtain a sequence of functions \( \{I_n\}_{n \in \mathbb{N}} \) which satisfies the following properties: for each \( k \in \mathbb{N} \),

\[
I_n = I_{n-1} = \cdots = I_k \text{ a.e. in } [-T - \eta^*, kb] \times \Omega, \forall n \geq k. \tag{21}
\]

For almost every \((t, x)\) in \([-T - \eta^*, + \infty) \times \Omega\), let

\[
\overline{T}(t, x) = \lim_{n \to +\infty} I_n(t, x).
\]

Since \( \overline{T}(t, x) = I_n(t, x) \) and

\[
I_n(t, x) = \int_{t-T}^{t} \alpha(t, x)(\rho(t, x) - I_n(t, x)) G[I_n](t, x) P(t, \tau) \, d\tau \tag{22}
\]

for almost every \((t, x)\) in \([-T - \eta^*, nb] \times \Omega\), we conclude easily that \( \overline{T} \) satisfies (13) and that it is a solution generated by \( I_0 \).

Under certain conditions, the previous existence result can be improved and we can also establish that the solutions decay exponentially to zero. This is the aim of the following theorem.

Let \( r(t, x) \) be the function defined by

\[
r(t, x) = \int_{t-T}^{t} \alpha(t, x) P(t, \tau) \, d\tau. \tag{23}
\]

**Theorem 3.3** We assume that \( \eta^* > 0 \) and \( \rho \in L^1(\Omega) \) is such that \( \rho(x) \geq 0 \) and \( \int_{\Omega} \rho(x) \, dx = 1 \). If \( r(t, x) \leq 1 \) for all \( t \geq 0 \), a.e. in \( \Omega \), then for each \( I_0 \in \mathcal{E} \) such that 0 \leq I_0(t, x) \leq \rho(x), \text{ a.e. in } Q, \) equation (8) has a unique solution \( \overline{T} \in \mathcal{E} \) generated by \( I_0 \). Moreover, if there exists \( \theta < 1 \) such that

\[
r(t, x) \leq \theta \text{ a.e. in } Q, \tag{24}
\]

then there exist positive constants \( C \) and \( L \) such that \( \overline{T}(t, x) \leq C e^{-Lt} \rho(x), \text{ for all } t \geq -T - \eta^* \) a.e. in \( \Omega \).

**Proof** The uniqueness follows immediately from Theorem 3.2. The proof of existence of solutions is based on the same three-steps arguments considering, for \(-T - \eta^* < a < b, \) the set

\[
\mathcal{F}_{a, b}(I_0) = \{ I \in C([-T - \eta^*, b]; L^1(\Omega)) ; 0 \leq I(t, x) \leq \rho(x) \text{ a.e. in } Q , \}
\]

\[
I = I_0 \text{ a.e. in } [-T - \eta^*, a] \times \Omega \}.
\]
which is a closed subset of the Banach space $C([-T - \eta^*, b]; L^1(\Omega))$, and the operator
$\Phi : \mathcal{F}_{a,b} \to \mathcal{F}_{a,b}$ defined as in (15) by $\Phi[I](t, x) = I_0(t, x)$ a.e. in $[-T - \eta^*, a] \times \Omega$ and

$$\Phi[I](t, x) = \int_{t-T}^{t} \alpha(\tau, x)(\rho(x) - I(\tau, x))G[I](\tau, x)P(t, \tau) \, d\tau$$

a.e. in $(a, b] \times \Omega$.

It is easily seen from (11) and (12) that $0 \leq G[I](t, x) \leq 1$ a.e. in $Q$ if $I \in \mathcal{F}_{a,b}$. The condition $r(t, x) \leq 1$ implies that $\mathcal{F}_{a,b}(I_0)$ is invariant for the operator $\Phi$, i.e., $\Phi[I] \in \mathcal{F}_{a,b}(I_0)$ if $I \in \mathcal{F}_{a,b}(I_0)$. Then, we can repeat the steps 2 and 3 in the previous theorem to obtain a solution $\mathcal{T} \in \mathcal{E}$ generated by $I_0$ satisfying

$$0 \leq \mathcal{T}(t, x) \leq \rho(x), \quad \text{a.e. in } Q,$$

where the function $\mathcal{T}(t, x)$ is obtained as a pointwise limit of a sequence $\{I_n\}$ satisfying (21).

On the other hand, remembering that $0 \leq p(x, y) \leq 1$ and denoting $\delta = T + \eta^*$, we obtain from (22) and the definition of $\mathcal{T}$,

$$\mathcal{T}(t, x) \leq r(t, x)\rho(x)||\mathcal{T}||_{C([-\delta - T, T]; L^1(\Omega))} \leq r(t, x)\rho(x)||\mathcal{T}||_{C_1([t-\delta, +\infty); L^1(\Omega))}. \quad (25)$$

Let us define the function

$$\psi(t) = ||\mathcal{T}||_{C_1([t-\delta, +\infty); L^1(\Omega))}.$$

If (24) holds, we obtain the following inequality by integrating on $\Omega$ both sides of (25),

$$\varphi(t) \leq \theta \max\{\varphi(s); s \geq t - \delta\} = \theta \psi(t), \quad \forall t \geq 0, \quad (26)$$

where $\varphi(t) = \int_{\Omega} \mathcal{T}(t, x) \, dx$. Since $\psi(t)$ is a decreasing function, we get easily from (26) that $\psi(t) \leq \theta \psi(t - \delta), \forall t \geq \delta$. In particular

$$\psi(n\delta) \leq \theta^n \psi(0), \quad \forall n \in \mathbb{N}. \quad (27)$$

Now, if $t \in [(n-1)\delta, n\delta)$, then we obtain from (25) and (27),

$$\mathcal{T}(t, x) \leq \theta \psi(t)\rho(x) \leq \theta \psi((n-1)\delta)\rho(x) \leq \theta^n \psi(0)\rho(x)$$

and the conclusion follows with

$$-L = \frac{\ln \theta}{b} \quad \text{and} \quad C = ||I_0||_{\mathcal{E}} \geq \psi(0) \quad \blacksquare$$
4. Existence of nontrivial stationary solutions

In this section we consider the existence of nontrivial stationary solutions for equation (8). More precisely, assuming that \( P(t, \tau) = \phi(t - \tau) \) and that \( \rho \in L^1(\Omega), \quad \alpha \in L^\infty(\Omega) \) do not depend on \( t \), we prove that, under certain conditions, there exists \( I \in L^1(\Omega), I \neq 0 \), such that

\[
I(x) = \alpha_T(x)(\rho(x) - I(x)) \int_\Omega p(x, y)I(y)\,dy,
\]

where \( \alpha_T(x) = \alpha(x) \int_0^T \phi(s)\,ds \).

As in the simpler homogeneous mixing case considered in [4], stationary solutions can be easily and explicitly obtained in certain particular cases. For instance, in the case that \( \alpha_T(x) = \alpha_T \) and \( p(x, y) = p_0 \) are constant functions, then the condition (2) implies that \( p_0 = 1 \) and it is easily seen that

\[
I_\infty(x) = \frac{\alpha_T - 1}{\alpha_T} \rho(x)
\]

is a solution of (28).

In order to prove the existence of nontrivial stationary solutions in a more general case, we first observe that (28) may be written as

\[
I(x) = \left( \frac{\alpha_T(x)c[I](x)}{1 + \alpha_T(x)c[I](x)} \right) \rho(x),
\]

where \( c[I] \) is the operator defined by

\[
c[I](x) = \int_\Omega p(x, y)I(y)\,dy.
\]

Therefore, we search for stationary solutions of (28) in the form \( I(x) = q(x)\rho(x) \), with \( q \in L^\infty(\Omega) \), by looking through fixed points of

\[
\Psi[q] = \frac{\alpha_T \beta[q]}{1 + \alpha_T \beta[q]},
\]

where \( \beta[q] \) is the integral operator defined by

\[
\beta[q](x) = \int_\Omega p(x, y)\rho(y)q(y)\,dy.
\]

Because of (11)-(i,ii), we consider the following numbers:

\[
\begin{align*}
p_* &= \text{ess inf} \{ p(x, y) ; x, y \in \Omega \}
p^* &= \text{ess sup} \{ p(x, y) ; x, y \in \Omega \}
\alpha_* &= \text{ess inf} \{ \alpha_T(x) ; x \in \Omega \}
\alpha^* &= \text{ess sup} \{ \alpha_T(x) ; x \in \Omega \}
\end{align*}
\]

\( R. \, Cipolatti \text{ and J.L. Gondar } \)
For $\varepsilon \geq 0$ we consider the set
\[ A_\varepsilon = \{ q \in L^\infty(\Omega) ; \varepsilon \leq q(x) \leq 1 \text{ a.e. in } \Omega \}, \tag{31} \]
which is a nonempty closed subset of $L^\infty(\Omega)$.

**Theorem 4.1** With the notation (30) and (31), we have:

(a) if $0 \leq \varepsilon < (\alpha_0 p_\ast - 1)/\alpha_0 p_\ast$, then $A_\varepsilon$ is invariant for the operator $\Psi$;
(b) if $0 < \alpha^* p^* < (\alpha_0 p_\ast)^2$, then there exists a unique $\bar{q} \in L^\infty(\Omega)$ satisfying
\[ \frac{\alpha_0 p_\ast - 1}{\alpha_0 p_\ast} \leq \bar{q}(x) \leq 1 \text{ in } \Omega \]
and such that $I^\infty(x) = \bar{q}(x)\rho(x)$ is a solution of (4.1).

**Proof** Since
\[ \Psi[q](x) \geq \varepsilon \iff \alpha_T(x)\beta[q](x) \geq \varepsilon/(1 - \varepsilon) \]
and since $\int_\Omega \rho(y) \, dy = 1$, we obtain
\[ q \geq \varepsilon \Rightarrow \beta[q](x) \geq p_\ast\varepsilon. \]
Then, it is sufficient to have $\varepsilon \leq (\alpha_0 p_\ast - 1)/\alpha_0 p_\ast$ in order to obtain $\Psi[q] \in A_\varepsilon$.

On the other hand, if $q_1, q_2 \in A_\varepsilon$ then, for $i = 1, 2$, $\beta[q_i](x) \geq \varepsilon p_\ast$. Therefore,
\[ \|\Psi[q_1] - \Psi[q_2]\|_{L^\infty} \leq \frac{\alpha^* p^*}{(1 + \alpha_0 p_\ast\varepsilon)^2}\|q_1 - q_2\|_{L^\infty}. \]
If $\alpha^* p^* < (\alpha_0 p_\ast)^2$, we can choose $\varepsilon > 0$ such that
\[ \frac{\sqrt{\alpha_T p^*} - 1}{\alpha_T p_\ast} < \varepsilon \leq \frac{\alpha_T p_\ast - 1}{\alpha_T p_\ast}. \]
With this choice, the inequality on the right assures that $A_\varepsilon$ is invariant for $\Psi$ and the one on the left assures that $\Psi$ is a contraction in $A_\varepsilon$. Hence, the Banach fixed point theorem can be applied and the proof is achieved.

**5. Numerical results and conclusions**

In order to perform numerical experiments to illustrate the salient features of the time evolution process, we consider the one dimensional case with $\Omega = [0, L]$, $L > 0$, and the simplest case of constant frequency $N_r$ and a $t$-constant infection-rate function $g(t, x) = I_0(x)/L$ for $(t, x) \in [-T - \eta^*, 0] \times [0, L]$, where $I_0$ represents the density of infective computers at $t = 0$. Furthermore, the probability of a computer infected at $\tau$ to remain infectious at $t$ was chosen as $P(t, \tau) = \phi(t - \tau)$, where $\phi$ is the characteristic function of the interval $[0, T)$, i.e., $\phi(s) = 1$ if $s \in [0, T)$ and $\phi(s) = 0$ elsewhere.
The first six experiments (figures 1a–3b) concern the case of a homogeneous distribution of computers on \( \Omega \), given by \( \rho(x) = 1/L \). The weight function \( p(x,y) \) is chosen as

\[
p(x,y) = C_v \left( 1 - \frac{1}{L v |x-y|^v} \right), \quad v > 0,
\]

where \( C_v \) must be defined by

\[
C_v = \frac{(v+1)(v+2)}{(v+1)(v+2) - 2}
\]

in order to satisfy the condition (2).

The other experiments concern the simulation of the dynamics in the presence of nonhomogeneity (which we call \textit{regional poles of softwares}). This means the existence of a subset \( \Omega_0 \subset \Omega \) with a greater concentration of computers and software resources.

Figures 1a and 1b show typical “slow infective rate cases”: \( r(t,x) = \alpha T < 1 \), for which the solutions decay exponentially to zero (see Theorem (3.3)). As it was shown in the previous section, the stationary solution \( I_\infty(x) = 0 \) is stable for \( r(t,x) = \alpha T < 1 \). This means that when such a condition holds, the introduction of a few infected computers into an infective-free population will not give rise to an epidemic outbreak and also no endemic situations will be developed, i.e., the infection will vanish along the time.

On the other hand, figures 2a–3b show the more complex situation of typical “fast infective rate cases” \( \alpha T > 1 \). In each case, an equilibrium point \( I_\infty(x) \) seems to be attained for \( t \to +\infty \), characterizing an endemic situation. Although it was proved in Theorem (4.1) the existence of stationary solutions only for \( p_s > 0 \), the numerical experiments indicate that they do exist even if \( p_s = 0 \). Moreover, even though it was not proved in the previous sections, they also indicate that the stationary solutions \( I_\infty(x) \) should be stable.

The other two experiments concern the simulation of the dynamics in the presence of \textit{regional poles of softwares}. This means the existence of a subset \( \Omega_0 \subset \Omega \) with a greater concentration of computers and/or software resources. In dealing with the existence of such a pole at \( \Omega_0 \), we assume the infection process to be directed, mainly, from inside the pole to its outside. Such a hypothesis is supported by the fact that a pole should be, in principle, a natural regional reference for recording processes and, for this reason, it also represents a contamination source from where virtual infections are spread over the whole remaining region. To illustrate this directional process we consider the particular case, where the weight function \( p(x,y) \) is a step function. More precisely, we consider the simple case where \( p \) has the following form:

\[
p = c_1 \Phi_{\Omega \times (\Omega \setminus \Omega_0)} + c_2 \Phi_{(\Omega \setminus \Omega_0) \times (\Omega \setminus \Omega_0)} + c_3 \Phi_{(\Omega \setminus \Omega_0) \times \Omega} + c_4 \Phi_{\Omega_0 \times \Omega_0},
\]

where \( \Phi_{A \times B} \) denotes the product measure on \( A \times B \).
Figure 1. The solutions $I(t, x)$ in a slow infective rate case: $\alpha = 0.4$. We consider $g(t, x) = I_0(x)/T$ for $t \in [-T - \eta^*, 0]$, where, in case (a) $I_0(x) = 4x(L - x)$, and in case (b) $I_0(x) = x$. Notice that in this case, the infection process affecting the system tends to disappear for $t$ large enough, in agreement with the previous results (see Theorem (3.3)).
Figure 2. The solutions $I(t,x)$ in a fast infective rate case. Here $\alpha = 1.2$ with $I_0(x)$ the same as in figure 1. An equilibrium endemic situation ($I_e(x) = (\alpha T - 1)/\alpha T r_0(x)$) is attained when $t \to \infty$, as can be seen from the figures. This last behavior seems to be a general one for $\alpha T > 1$, because using different parameters in numerical calculations we have always obtained such a result.
Dynamics of a virtual virus infection process

Figure 3. The same as in figure 2, with $\alpha = 2.4$. 
Figure 4. The solutions $I(t,x)$ in the presence of a pole. Here $\alpha = 0.45$ with $I_0(x) = x(2 - x)$ in the case (a), and $\alpha = 1.2$ with $I_0(x) = x/2$ in the case (b).
where \( \Phi_A \) denotes the characteristic function of the set \( A \subset \Omega \times \Omega \) and \( c_i \geq 0 \), \( i = 1, \ldots, 4 \) are the corresponding weights. We can describe that the infection process is mainly directed from inside the pole to its outside assuming that \( c_3 > c_i \), for \( i = 1, 2, 4 \).

In the experiments, we consider once more the simplest one dimensional case \( \Omega = [0, L] \) with a uniform distribution of computers \( (\rho(t, x) = 1/L) \) and a pole of softwares located at \( x_0 = [x_0 - \epsilon, x_0 + \epsilon] \), choosing the parameters in such a way that the condition (2) be verified.

In the figures 1–3 below we consider \( \Omega = [0, 1] \), \( \eta(x, y) = \eta_x = \eta^* = 1 \), \( T = 2 \), the weight function \( p(x, y) \) given by (32) with \( v = 1 \), and \( P(t, \tau) = \phi(t - \tau) \), where \( \phi(\xi) \) is the characteristic function of \( [0, T] \).

In the figures 4 below we consider \( \Omega = [0, 2] \), \( \eta(x, y), T, P(t, \tau) \) as before, and the weight function \( p(x, y) \) given by (33) with the following choice of parameters: \( x_0 = 1, \epsilon = 1/2, c_1 = 0, c_2 = c_4 = 1 \) and \( c_3 = 2 \).

References