NONLINEAR BOUNDARY FEEDBACK STABILIZATION FOR SCHRÖDINGER EQUATIONS

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Abstract: In this paper we consider the existence, uniqueness and the asymptotic behavior of the solutions of Schrödinger equations in bounded domains with nonlinear dissipative boundary conditions; the existence and uniqueness are proved by means of nonlinear semigroup theory and the asymptotic behavior is obtained by establishing decay rates for the energy.

1. Introduction

The problem of proving uniform decay rates for solutions of systems of evolution equations with dissipation at the boundary has been treated by several authors. Indeed, in the case of wave or plate equations we can mention J. Lagnese [5], I. Lasiecka [6], V. Komornik and E. Zuazua [3,4], J. L. Lions [7], E. Zuazua [10,11], among others.

Very few is known for the Schrödinger equation. To our knowledge, exponential decay was obtained by E. Machtyngier and E. Zuazua [9] and E. Machtyngier [8] when the boundary dissipation is linear.

In this paper we consider the problem of nonlinear feedback stabilization for Schrödinger equation. More precisely, we consider the following problem.

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\[
\begin{cases}
  iy_t + \Delta y = 0 & \text{in } \Omega \times (0, +\infty), \\
  \frac{\partial y}{\partial \nu} + q(y_t) = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\
  y = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\
  y(0) = y_0 & \text{in } \Omega,
\end{cases}
\] (1.1)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \) \((\Gamma_i \neq \emptyset, i = 0, 1)\) and \( q \) is a nonlinear function satisfying some general properties (see (1.3)–(1.5)).

Before describing the organization of the paper and present the main results, we introduce the basic notation to be used from now on.

We consider the space \( L^2(\Omega) \) of complex valued functions on \( \Omega \). It is a real Hilbert space when endowed with the scalar product

\[
(u; v)_2 = \Re \int_{\Omega} u(x) \overline{v(x)} \, dx,
\]

with the corresponding norm

\[
\|u\|_2^2 = (u; u)_2.
\]

We consider also the Sobolev space \( H^1(\Omega) \) endowed with scalar product

\[
(u; v)_{H^1} = (u; v)_2 + (\nabla u; \nabla v)_2.
\]

The natural space for problem (1.1) is

\[
V = \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_1 \}.
\]

When \( \Gamma_1 \) has nonempty interior in \( \partial \Omega \), we have Poincaré’s inequality

\[
\|u\|_2 \leq \alpha \|\nabla u\|_2, \quad \forall u \in V \tag{1.2}
\]

Thus, we may consider \( V \) endowed with the norm induced by the scalar product

\[
(u; v)_V = \Re \int_{\Omega} \nabla u(x) \cdot \overline{\nabla v(x)} \, dx,
\]

which, in \( V \), is equivalent to the norm of \( H^1(\Omega) \). As usual, we denote by \( V' \) the dual of \( V \).
We organize this paper in two sections. Section 2 is devoted to the existence of solutions of the Cauchy problem for the system (1.1). In this case we assume that

\[ q: H^{1/2}(\Gamma_0) \rightarrow H^{-1/2}(\Gamma_0) \]  

(1.3)

is the Fréchet derivative of some functional \( Q \) satisfying

\[ Q \in C^1(H^{1/2}(\Gamma_0); \mathbb{R}), \]  

(1.4)

\[ Q \] is convex.  

(1.5)

A particular example is given by \( q(v)(x) = a(x)g(v(x)), x \in \Gamma_0, \) with \( a \in L^\infty(\Gamma_0) \) and \( g:\mathbb{C} \rightarrow \mathbb{C} \) is of the form \( g(z) = f(|z|)\frac{z}{|z|} \), where \( f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is continuous and monotone and satisfies

\[ |f(s)| \leq c(1 + s)^\gamma, \]  

(1.6)

for some \( c > 0 \) and \( \gamma \in [0, \frac{N+1}{N-1}] \).

Our proof of existence and uniqueness of solution of (1.1) is inspired by I. Lasiecka in [6] (see also [5; p. 29 and 100]). We consider (1.1) as a nonlinear Cauchy problem in \( V \) of the form

\[
\begin{align*}
  y_t &= C(y) \\
  y(0) &= y_0
\end{align*}
\]  

(1.7)

where \( C: D(C) \subset V \rightarrow V \) is defined by

\[ C(y) = i\Delta y, \quad \forall y \in D(C), \]  

(1.8)

\( D(C) \) being a dense set in \( V \) (see (2.6)).

The main result of section 2 is then the following.

**Theorem 1.1.** Assume that \( \Gamma_1 \) has nonempty interior in \( \partial \Omega \). If (1.3)–(1.5) hold, then \( C \) generates a nonlinear semigroup of contractions \( \{S(t)\}_{t \geq 0} \) in \( V \). Furthermore, if \( y_0 \in D(C) \), then \( y(t) = S(t)y_0 \in D(C) \) for all \( t > 0 \) and we have

\[ y \in W^{1,\infty}(0, \infty; V), \]

\[ y_t = C(y) \text{ a.e. in } [0, -\infty). \]
In section 3 we consider the stabilization properties for the solutions of (1.1). We assume that $\Omega \subset \mathbb{R}^N$, with $N \leq 3$ is a domain of class $C^3$ satisfying the following geometrical properties: there exists $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ such that
\begin{align*}
\Gamma_0 &= \{x \in \partial \Omega \mid (x - x_0) \cdot \nu(x) > 0\}, \\
\Gamma_1 &= \{x \in \partial \Omega \mid (x - x_0) \cdot \nu(x) \leq 0\},
\end{align*}
(1.9)
(1.10)
where $\nu(x)$ denotes the unit outward normal vector and “·” is the usual scalar product in $\mathbb{R}^N$.
We set $m(x) = x - x_0$ and we assume that the nonlinear term is
\begin{equation}
q(u)(x) = (m(x) \cdot \nu(x))g(u(x)), \quad x \in \Gamma_0,
\end{equation}
(1.11)
where $g:\mathbb{C} \to \mathbb{C}$ is of the form
\begin{equation}
g(z) = f(|z|)\frac{z}{|z|}, \quad z \neq 0,
\end{equation}
(1.12)
$f: \mathbb{R}^+ \to \mathbb{R}^+$ being continuous and monotone such that (1.6) holds and
\begin{equation}
f(s) > 0 \text{ if } s > 0.
\end{equation}
(1.13)
To obtain the stabilization results for problem (1.1), we define the energy of a solution $y = y(x, t)$ of (1.1) as
\begin{equation}
E(t) = \frac{1}{2} \int_{\Omega} |\nabla y(x, t)|^2 \, dx = \frac{1}{2} \|y(t)\|^2_V, \quad \forall t \geq 0
\end{equation}
(1.14)
Multiplying (1.1) by $\overline{y}_t$, integrating by parts and taking the real part, we formally obtain that
\begin{equation}
E'(t) = \frac{dE(t)}{dt} = -\int_{\Gamma_0} (m \cdot \nu)g(y_t)\overline{y}_t \, d\Gamma,
\end{equation}
(1.15)
where $d\Gamma$ is the surface measure on the boundary $\Gamma$. Taking into account (1.12) and (1.13), identity (1.15) shows that the boundary conditions of system (1.1) are dissipative in the space $V$.
As mentioned above, E. Machtyngier and E. Zuazua [9] (and E. Machtyngier [8]) have proved the exponential decay of the energy $E$ for dimensions $N \leq 3$ in the case where $g(z) = z$. They used the method based on the construction of energy
functionals, developed by V. Komornik and E. Zuazua [3,4] for the wave equation. This method was adapted by E. Zuazua in [10] to obtain decay rates for the wave equations with nonlinear dissipative boundary conditions. We use these techniques to estimate the decay rate for the energy of the solution to problem (1.1) in the case where the dissipation on the boundary is nonlinear (i.e. $g(z)$ is nonlinear). In this case we assume that there exist $p \geq 1$ and $0 < \lambda \leq p$ such that, for $s < 1$,

$$C_1 \min\{s, s^p\} \leq f(s) \leq C_2 s^\lambda,$$

(1.16)

$C_1$ and $C_2$ being positive constants.

With these assumptions, we have the following decay rates for the energy associated to problem (1.1):

**Theorem 1.2.** Assume that $N \leq 3$, (1.9)–(1.13) and (1.16). For any $y_0 \in D(C)$ the solution $y(t)$ of (1.1) satisfies:

a) If $\lambda = p = 1$, there exist constants $c > 1$ and $\gamma > 0$ such that

$$E(t) \leq cE(0) \exp(-\gamma t), \quad t \geq 0. \quad (1.17)$$

b) If $\lambda \geq 1$, $p > 1$, there exist constants $c > 1$, $\mu_{p,\lambda} > 0$ (with $\mu_{p,\lambda}$ depending on $E(0)$) such that

$$E(t) \leq cE(0) \left(1 + t\mu_{p,\lambda}\right)^{-\frac{2}{(p-1)}} \quad \forall t \geq 0. \quad (1.18)$$

c) If $\lambda \leq 1$, $p > 1$, there exist constants $c > 1$, $\mu_{p,\lambda} > 0$ (with $\mu_{p,\lambda}$ depending on $E(0)$) such that

$$E(t) \leq cE(0) \left(1 + t\mu_{p,\lambda}\right)^{-\frac{2\lambda}{(p+1)-2\lambda}} \quad \forall t \geq 0. \quad (1.19)$$

**Remarks.**

1) The assumption $N \leq 3$ is due to the use of Grisvard’s inequality which has been proved only for dimensions less or equal 3 (see [2], [10]).
2) Let \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) defined by either \( f(s) = \begin{cases} s^{1/2} & \text{if } s < 1 \\ 1 & \text{if } s \geq 1 \end{cases} \) or else \( f(s) = s^p \) for some \( p \leq \frac{N+1}{N-1} \), and let \( F(s) = \int_0^s f(\xi) \, d\xi \). Then \( Q \) defined by

\[
Q(u) = \int_{\Gamma_0} m(x) \cdot \nu(x) F(|u(x)|) \, d\Gamma
\]

verifies the assumptions of Theorem 1.2 with \( g(z) = f(|z|) \frac{z}{|z|} \).

2. The existence of solutions

As we have mentioned above, in order to prove Theorem 1.1, we introduce the following nonlinear Cauchy problem on \( V \)

\[
\begin{cases}
y_t = \mathcal{C}(y), \\
y(0) = y_0,
\end{cases}
\]

(2.1)

where \( \mathcal{C}: D(\mathcal{C}) \subset V \to V \) is defined as follows.

We consider the operator \( A \) in \( V \) defined by

\[
D(A) = \{ y \in V \mid \Delta y \in L^2(\Omega) \text{ and } \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma_0 \}
\]

and \( Ay = \Delta y \) for all \( y \in D(A) \), where \( \frac{\partial y}{\partial \nu}|_{\Gamma_0} \) is defined by using Green’s formula.

More precisely, \( y \in D(A) \) if and only if \( y \in V \) such that \( \Delta y \in L^2(\Omega) \) and

\[
(\nabla y; \nabla u)_2 + (\Delta y; u)_2 = 0 \quad \forall u \in V
\]

Consider now the Neumann operator \( \mathcal{N} \in \mathcal{L}(H^{-1/2}(\Gamma_0); V) \) defined by

\[
\mathcal{N} \gamma = v \quad \iff \quad \begin{cases}
\Delta v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \Gamma_1, \\
\frac{\partial v}{\partial \nu} = -\gamma & \text{on } \Gamma_0.
\end{cases}
\]

By composing \( A \) and \( \mathcal{N} \) we may define the operator \( B \). More precisely,

\[
\begin{cases}
B: H^{-1/2}(\Gamma_0) \to V', \\
B(u) = \tilde{A} \mathcal{N} u,
\end{cases}
\]

(2.2)

where \( \tilde{A} \) is the self-adjoint extension of \( A \) to \( V \).

The adjoint of \( B \) is then \( B^* = \mathcal{N}^* \tilde{A} \), where \( \mathcal{N}^* \) is the adjoint of \( N \) with respect to \( L^2(\Omega) \). The operator \( B^* \) has the following properties:

\[
B^* \in \mathcal{L}(V; H^{1/2}(\Gamma_0)),
\]

(2.3)

\[
B^* v = v|_{\Gamma_0}.
\]

(2.4)
As a consequence of the trace theorem, we have also

\[ B^* \text{ is surjective.} \quad (2.5) \]

Finally, let \( C \) be the nonlinear operator defined by

\[
D(C) = \{ y \in V \mid \Delta y \in V, \ y - Nq(B^*(i\Delta y)) \in D(A) \},
\]

\[
C(y) = i\Delta y, \quad \forall y \in D(C),
\]

where \( q \) satisfies (1.3)–(1.5). Then we can rewrite (1.1) as (2.1). In particular we have:

**Lemma 2.1.** \( D(C) \) is dense in \( V \).

**Proof:** Let

\[
\mathcal{X} = \{ y \in H^2(\Omega) \mid \Delta y \in H^1_0(\Omega), y|_{\Gamma_1} = 0, \frac{\partial y}{\partial \nu}|_{\Gamma_0} = 0 \}.
\]

Then \( \mathcal{X} \) is a subspace of \( D(A) \) which is dense in \( V \). Indeed, let \( u \in V \) such that \( (u; y)_V = 0 \) for all \( y \in \mathcal{X} \). Since \( (w; y)_V = (\Delta y; u)_2 \), it is sufficient to show that \( \{ \Delta y \mid y \in \mathcal{X} \} \) is dense in \( L^2(\Omega) \). To see that, let \( w \in L^2(\Omega) \) and consider \( \{ w_n \}_{n \geq 1} \subset H^1(\Omega) \) such that \( w_n \rightharpoonup w \) in \( L^2(\Omega) \). For each \( n = 1, 2, \ldots \), we define \( y_n \) by

\[
\left\{ \begin{array}{l}
\Delta y_n = w_n \quad \text{in } \Omega, \\
y_n = 0 \quad \text{on } \Gamma_1, \\
\frac{\partial y_n}{\partial \nu} = 0 \quad \text{on } \Gamma_0.
\end{array} \right.
\]

It follows that \( y_n \in \mathcal{X} \) and \( \Delta y_n \rightharpoonup w \) in \( L^2(\Omega) \).

We define now \( \mathcal{X}_1 = \mathcal{X} + Nq(0) \). Since \( Nq(0) \in V \), we have \( \mathcal{X}_1 \subset V \) and so \( \mathcal{X}_1 = V \).

Hence the result, since \( \mathcal{X}_1 \subset D(C) \).

We are now in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We proceed in three steps.

**Step 1. The operator \(-C\) is monotone in \( V \).**

First of all we note that, by (2.4) and (2.6),

\[
\frac{\partial y}{\partial \nu}|_{\Gamma_0} = -q(B^*(i\Delta y)) \quad \forall y \in D(C).
\]

\[
(2.8)
\]
Let $y_i \in D(C), i = 1, 2$. Since $\Delta y_i \in V$, we have $\Delta y_i|_{\Gamma_1} = 0$. Then
\[
(C(y_1) - C(y_2); y_1 - y_2)_V = -\left(i\Delta(y_1 - y_2); -\Delta(y_1 - y_2)\right)_2 + \langle \frac{\partial y_1}{\partial \nu} - \frac{\partial y_2}{\partial \nu}; B^*(i\Delta(y_1 - y_2)) \rangle,
\]
where $\langle ; \rangle$ denotes the $H^{-1/2}(\Gamma_0) - H^{1/2}(\Gamma_0)$ duality pairing.

Since $\left(i\Delta(y_1 - y_2); \Delta(y_1 - y_2)\right)_2 = 0$, it follows from (2.8) that
\[
(C(y_1) - C(y_2); y_1 - y_2)_V = -\langle q(B^*(i\Delta y_1)) - q(B^*(i\Delta y_2)); B^*(i\Delta y_1 - i\Delta y_2) \rangle
\]
and the conclusion follows from (1.3)–(1.5).

**Step 2. The operator $-C$ is maximal monotone.**

Let $f \in V$ and consider the equation
\[
y - C(y) = f.
\]
(2.9)

In order to prove that (2.9) has a solution $y$ in $D(C)$, let us consider the operator $M: V \to D(C) \subset V$ defined by
\[
Mu = y \iff \begin{cases} -\Delta y = iu & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_1, \\ \frac{\partial y}{\partial \nu} = -q(B^*u) & \text{on } \Gamma_0. \end{cases}
\]
(2.10)

From the definitions of $C$ and $M$ we can rewrite (2.9) as
\[
u - Mu = -f
\]
(2.11)

$M$ satisfies the following properties:

a) $-M$ is monotone in $V$.

Indeed, by (2.10)
\[
(Mu_1 - Mu_2; u_1 - u_2)_V = -\left(i\Delta(y_1 - y_2); i\Delta(y_1 - y_2)\right)_2 + \langle \frac{\partial y_1}{\partial \nu} - \frac{\partial y_2}{\partial \nu}; B^*(i\Delta y_1 - i\Delta y_2) \rangle
\]
\[
= -\langle q(B^*(i\Delta y_1)) - q(B^*(i\Delta y_2)); B^*(i\Delta y_1 - i\Delta y_2) \rangle,
\]
and the conclusion follows from (1.3)–(1.5).

b) $M$ is continuous in $V$
Indeed, the operator $\mathcal{M}: L^2(\Omega) \times H^{-1/2}(\Gamma_0) \rightarrow V$ defined by

$$\mathcal{M}(f, \gamma) = y \iff \begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_1, \\ \frac{\partial y}{\partial \nu} = -\gamma & \text{on } \Gamma_0. \end{cases}$$

is linear and continuous. From (2.10) we have

$$Mu = \mathcal{M}\left( iu, q(B^*(u)) \right)$$

and the conclusion of b) follows easily. From a) and b) we have that $-M$ is maximal monotone (see [1], Proposition 2.4). Hence, for $f \in V$, (2.11) has a unique solution $u \in V$ and $y = Mu$ is a solution of (2.9).

Step 3. From Lemma 2.1 and steps 2 and 3, the operator $\mathcal{C}$ is the infinitesimal generator of a nonlinear semigroup $\{S(t)\}_{t \geq 0}$ on $D(\mathcal{C}) = V$. Hence the conclusion. □

3. The stabilization

In this section we prove Theorem 1.2. For this purpose we make use of a method based on the perturbation of the energy introduced by V. Komornik and E. Zuazua for the wave equation. In the context of Schrödinger’s equation, it was first used by E. Machtyngier and E. Zuazua in [9] when $g(s) = s$ (the linear problem). Before the proof of theorem 1.2, let us state the following useful inequality (see [2] and [10]).

Lemma 3.1. Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, a domain of class $C^3$. Then the following inequality holds

$$2\Re \int_{\Omega} \Delta y (m \cdot \nabla \eta) \, dx \leq (N - 2) \int_{\Omega} |\nabla y|^2 \, dx + 2\Re \int_{\Gamma} \frac{\partial y}{\partial \nu} m \cdot \nabla \eta \, d\Gamma$$

$$- \int_{\Gamma} (m \cdot \nu) |\nabla y|^2 \, d\Gamma,$$

for every $y \in V$ such that $\Delta y \in L^2(\Omega)$ and $\frac{\partial y}{\partial \nu} = (m \cdot \nu)v$ on $\Gamma_0$ with $v \in L^2(\Gamma_0)$.
Remarks.

1) In [3] and [4] inequality (3.1) was proved in the case where \( v \in H^{1/2}(\Gamma_0) \). In [10] it was extended to the case where \( v \in L^2(\Gamma_0) \).

2) Since \( y \in V \), we may replace in Lemma 3.1 the surface integrals on \( \Gamma \) by integrals on \( \Gamma_0 \). More precisely we have:

\[
2 \Re \int_\Omega \Delta y (m \cdot \nabla \overline{y}) \, dx \leq (N - 2) \int_\Omega |\nabla y|^2 \, dx + 2 \Re \int_{\Gamma_0} \frac{\partial y}{\partial \nu} m \cdot \nabla \overline{y} \, d\Gamma \\
- \int_{\Gamma_0} (m \cdot \nu) |\nabla y|^2 \, d\Gamma. \tag{3.1}
\]

Indeed, assume for simplicity that \( y \) is regular enough such that \( y = 0 \) on \( \Gamma_1 \). Then \( \nabla y = \frac{\partial y}{\partial \nu} \nu \); and so

\[
\frac{\partial y}{\partial \nu} m \cdot \nabla \overline{y} = (m \cdot \nu) |\frac{\partial y}{\partial \nu}|^2 = (m \cdot \nu) |\nabla y|^2 \text{ on } \Gamma_1.
\]

Since \( m \cdot \nu \leq 0 \) on \( \Gamma_1 \) we have

\[
2 \Re \int_{\Gamma} \frac{\partial y}{\partial \nu} m \cdot \nabla \overline{y} \, d\Gamma - \int_{\Gamma} (m \cdot \nu)|\nabla y|^2 \, d\Gamma \leq 2 \Re \int_{\Gamma_0} \frac{\partial y}{\partial \nu} m \cdot \nabla \overline{y} \, d\Gamma - \int_{\Gamma_0} (m \cdot \nu)|\nabla y|^2 \, d\Gamma.
\]

**Proof of theorem 1.2.** We proceed in several steps.

**Step 1.** Let \( y_0 \in D(C) \). From Theorem 1.1 the Cauchy problem (2.1) has a unique solution \( y \) such that \( y(t) \in D(C) \), for all \( t \geq 0 \) and (1.15) holds. Moreover, sup \( \|\nabla y(t)\|_V < +\infty \); and so sup \( \|y(t)\|_{H^3(\Omega)} < +\infty \). It follows from Sobolev imbedding theorem that \( y(t) \) is bounded in \( C^1(\overline{\Omega}) \), and we define \( B_0, B_1 \) by

\[
B_0 = \sup_{t \geq 0} \|y(t)\|_{L^\infty(\Gamma)} \\
B_1 = \sup_{t \geq 0} \|\nabla y(t)\|_{L^\infty(\Gamma)}
\]

**Step 2.** We define the function \( \rho \) as

\[
\rho(t) = (-iy, m \cdot \nabla y)_2, \quad \forall t \geq 0. \tag{3.2}
\]

From (1.2) and (1.14) we obtain

\[
|\rho(t)| \leq 2r\alpha E(t), \quad \forall t \geq 0, \tag{3.3}
\]

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where $r = \|m\|_{L^\infty(\Gamma)}$. Since $\rho'(t) = (-iy_t, m \cdot \nabla y)_2 + (-iy, m \cdot \nabla y_t)_2$, we have from (1.1), (1.11), (3.1) and the divergence theorem

$$
\rho'(t) \leq -4E(t) - \Re \int_{\Gamma_0} (m \cdot \nu)[(2m \cdot \nabla \bar{y} + N\bar{y})g(y_t) + \vert \nabla y \vert^2 + iy\bar{y}_t] \, d\Gamma. \tag{3.4}
$$

**Step 3.** We consider

$$
\Gamma_0^+(t) = \{x \in \Gamma_0 \mid |y_t(t, x)| \geq 1\}
$$

$$
\Gamma_0^-(t) = \{x \in \Gamma_0 \mid |y_t(t, x)| < 1\}
$$

The following estimates hold:

$$
\left| 2\Re \int_{\Gamma_0} (m \cdot \nu)g(y_t)m \cdot \nabla \bar{y} \, d\Gamma \right| \leq 2\mathcal{R}B_1 \int_{\Gamma_0^+(t)} (m \cdot \nu)|g(y_t)| \, d\Gamma 
$$

$$
+ r^2 \int_{\Gamma_0^-(t)} (m \cdot \nu)|g(y_t)|^2 \, d\Gamma \tag{3.5}
$$

$$
+ \int_{\Gamma_0^-(t)} (m \cdot \nu)|\nabla y|^2 \, d\Gamma,
$$

$$
\left| N\Re \int_{\Gamma_0} (m \cdot \nu)g(y_t)\bar{y} \, d\Gamma \right| \leq B_0N \int_{\Gamma_0^+(t)} (m \cdot \nu)|g(y_t)| \, d\Gamma
$$

$$
+ \frac{N}{2a} \int_{\Gamma_0^-(t)} (m \cdot \nu)|g(y_t)|^2 \, d\Gamma \tag{3.6}
$$

$$
+ \frac{Nra}{2} \int_{\Gamma_0^-(t)} |y|^2 \, d\Gamma,
$$

$$
\left| \Re \int_{\Gamma_0} i(m \cdot \nu)y\bar{y}_t \, d\Gamma \right| \leq B_0 \int_{\Gamma_0^+(t)} (m \cdot \nu)|y_t| \, d\Gamma + \frac{1}{2b} \int_{\Gamma_0^-(t)} (m \cdot \nu)|y_t|^2 \, d\Gamma
$$

$$
+ \frac{br}{2} \int_{\Gamma_0^-(t)} |y|^2 \, d\Gamma \tag{3.7}
$$

for $a, b > 0$. We observe that $\|y\|_{L^2(\Gamma_0)} \leq \delta\|\nu\|$ for some $\delta > 0$, and we choose $a$ and $b$ such that

$$
\frac{Nra}{2} + \frac{br}{2} = \frac{1}{\delta^2}.
$$

Hence, from (3.4)–(3.7) we obtain

$$
\rho'(t) \leq -2E(t) + c_1 \int_{\Gamma_0^+(t)} (m \cdot \nu)[|y_t| + |g(y_t)|] \, d\Gamma
$$

$$
+ c_2 \int_{\Gamma_0^-(t)} (m \cdot \nu)[|y_t|^2 + |g(y_t)|^2] \, d\Gamma. \tag{3.8}
$$

**Step 4.** Let

$$
\theta(\lambda) = \begin{cases} 
\frac{b+1}{2\lambda} - 1 & \text{if } 0 < \lambda < 1, \\
\frac{b-1}{2} & \text{if } \lambda \geq 1.
\end{cases}
$$
Since \(E'(t) \leq 0\), it follows from (3.3) that
\[
\frac{d}{dt} (E(t)^{\theta(\lambda)} \rho(t)) \leq -c_3(\lambda) E'(t) + E(t)^{\theta(\lambda)} \rho'(t),
\] (3.9)
where \(c_3(\lambda) = 2\alpha \theta(\lambda) E(0)^{\theta(\lambda)}\).

For \(\varepsilon > 0\) we define the function \(E_\varepsilon\) by
\[
E_\varepsilon(t) = (1 + \varepsilon c_3(\lambda)) E(t) + \varepsilon E(t)^{\theta(\lambda)} \rho(t)
\] (3.10)
Since the energy is nonincreasing, we obtain from (3.3) and (3.10)
\[
1/2 E(t)^{\theta(\lambda)+1} \leq E_\varepsilon(t)^{\theta(\lambda)+1} \leq 2 E(t)^{\theta(\lambda)+1},
\] (3.11)
provided we chose \(\varepsilon > 0\) such that
\[
\varepsilon E(0)^{\theta(\lambda)} \leq \frac{1 - 2^{-(\theta(\lambda)+1)^{-1}}}{2\alpha(\theta(\lambda) + 1)}.
\]

Taking derivatives in both sides of (3.10) and applying (3.9) we obtain
\[
E'_\varepsilon(t) \leq E'(t) + \varepsilon E(t)^{\theta(\lambda)} \rho'(t).
\] (3.12)

Merging (3.8) in (3.12), we obtain
\[
E'_\varepsilon(t) \leq E'(t) - 2\varepsilon E(t)^{\theta(\lambda)+1}
+ \varepsilon c_1 E(t)^{\theta(\lambda)} \int_{\Gamma_0^+} (m \cdot \nu)[|y_t| + |g(y_t)|] d\Gamma
+ \varepsilon c_2 E(t)^{\theta(\lambda)} \int_{\Gamma_0^+} (m \cdot \nu)[|y_t|^2 + |g(y_t)|^2] d\Gamma.
\] (3.13)

From the definition of \(\Gamma_0^+(t)\), (1.12) and (1.13) we get
\[
|g(y_t)| \leq g(y_t) \bar{y}_t \quad \text{on} \quad \Gamma_0^+(t),
\]
\[
|y_t| \leq \frac{1}{f(1)} g(y_t) \bar{y}_t \quad \text{on} \quad \Gamma_0^+(t).
\]

Therefore
\[
\int_{\Gamma_3^+} (m \cdot \nu)[|y_t| + |g(y_t)|] d\Gamma \leq (1 + 1/f(1)) \int_{\Gamma_0^+} (m \cdot \nu) g(y_t) \bar{y}_t d\Gamma.
\]
Choosing \(\varepsilon_1 > 0\) such that \(\varepsilon_1 c_1 E(0)^{\theta(\lambda)} (1 + 1/f(1)) < 1\), we obtain from (3.13) and (1.15)
\[
E'_\varepsilon(t) \leq -\int_{\Gamma_0^+(t)} (m \cdot \nu) g(y_t) \bar{y}_t d\Gamma - 2\varepsilon E(t)^{\theta(\lambda)+1}
+ \varepsilon c_2 E(t)^{\theta(\lambda)} \int_{\Gamma_0^+} (m \cdot \nu)[|y_t|^2 + |g(y_t)|^2] d\Gamma,
\] (3.14)
for all \(\varepsilon < \varepsilon_1\).
Step 5. At this step we must consider three cases:

Case 1. We assume \( \lambda = 1 \) and \( p = 1 \). In this case we have \( \theta(1) = 0 \). Choosing \( \varepsilon = \gamma < C_1/c_2(1 + C_2^2) \) in (3.14), we obtain from (1.16) that \( E'_\varepsilon(t) \leq -2\gamma E(t) \).

Therefore (1.17) follows from (3.11).

Case 2. Let \( \lambda \geq 1 \) and \( p > 1 \). From (1.16) and (3.14) we have

\[
E'_\varepsilon(t) \leq -\int_{\Gamma_0^-(t)} (m \cdot \nu)g(y_t) y_t d\Gamma - 2\varepsilon E(t)^{(p+1)/2} \\
+ \varepsilon c_2(1 + C_2^2) E(t)^{(p-1)/2} \int_{\Gamma_0^-(t)} (m \cdot \nu)|y_t|^2 d\Gamma.
\]

Let \( I = \int_{\Gamma_0^-(t)} (m \cdot \nu) d\Gamma \). Applying the Hölder inequality in the last term of (3.15) we obtain

\[
E'_\varepsilon(t) \leq -\int_{\Gamma_0^-(t)} (m \cdot \nu)g(y_t) y_t d\Gamma - 2\varepsilon E(t)^{(p+1)/2} \\
+ \varepsilon c_2(1 + C_2^2) E(t)^{(p-1)/2} (p+1) \\
\left( \int_{\Gamma_0^-(t)} (m \cdot \nu)|y_t|^{p+1} d\Gamma \right)^{2/(p+1)}.
\]

Using Young inequality in the last term and taking \( \varepsilon \) such that

\[
(\varepsilon/C_1)^{2/(p-1)}(c_2(1 + C_2^2))^{(p+1)/(p-1)} I \leq 1,
\]

we have from (1.16) and (3.11)

\[
E'_\varepsilon(t) \leq -\varepsilon E(t)^{(p+1)/2} \leq -\frac{\varepsilon}{2} E_\varepsilon(t)^{(p+1)/2}.
\]

Solving (3.16) we conclude that

\[
E_\varepsilon(t) \leq E_\varepsilon(0) \left( 1 + \frac{\varepsilon(p-1)}{4} E_\varepsilon(0)^{(p-1)/2} t \right)^{-2/(p-1)}
\]

Using again (3.11) in the above inequality we obtain

\[
E(t) \leq 2^{4/(p+1)} E(0) \left( 1 + \mu_{p,\lambda}(E(0)) t \right)^{-2/(p-1)},
\]

with \( \mu_{p,\lambda}(s) = \varepsilon(p-1)2^{-(3p+1)/(p+1)}s^{(p-1)/2} \).

Case 3. Let \( \lambda < 1 \) and \( p \geq 1 \). We rewrite (3.14) as

\[
E'_\varepsilon(t) \leq -\int_{\Gamma_0^-(t)} (m \cdot \nu)g(y_t) y_t d\Gamma - 2\varepsilon E(t)^{(p+1)/2\lambda} \\
+ \varepsilon c_2(1 + C_2^2) E(t)^{(p+1)/2\lambda-1} \int_{\Gamma_0^-(t)} (m \cdot \nu)|y_t|^{2\lambda} d\Gamma.
\]

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Again, we apply H"older in the last term to obtain
\[
E'_\varepsilon(t) \leq -\int_{\Gamma_0(t)} (m \cdot \nu) g(y_t) \overline{g_t} \, d\Gamma - 2\varepsilon E(t)^{(p+1)/2\lambda} + \varepsilon c_2 (1 + C_2^2) E(t) \frac{p+1}{2\lambda-1} (1 + c_2^2) \left( \int_{\Gamma_0(t)} |m \cdot \nu| |y_t|^{p+1} \, d\Gamma \right)^{\frac{2\lambda}{p+1}}.
\]
Taking \( \varepsilon \) such that
\[
\left( \varepsilon/C_1 \right)^{(2\lambda/p+1-2\lambda)} (c_2 (1 + C_2^2))^{(p+1)/p+1-2\lambda} \leq 1
\]
and using the Young inequality, we obtain from (1.16) and (3.11)
\[
E'_\varepsilon(t) \leq -\varepsilon E(t)^{(p+1)/2\lambda} \leq -\frac{\varepsilon}{2} E(0)^{(p+1)/2\lambda}.
\]
Hence,
\[
E_\varepsilon(t) \leq E_\varepsilon(0) \left( 1 + \frac{\varepsilon(p+1-2\lambda)}{4\lambda} E_\varepsilon(0)^{(p+1-2\lambda)/2\lambda} \right)^{-2\lambda/(p+1-2\lambda)},
\]
and we conclude by (3.11) that
\[
E(t) \leq 2^{\lambda/(p+1)} E(0) \left( 1 + \mu_{p,\lambda}(E(0)) t \right)^{-2\lambda/(p+1-2\lambda)}
\]
with \( \mu_{p,\lambda}(s) = \varepsilon \lambda^{-1} (p + 1 - 2\lambda) 2^{-(3p+3-2\lambda)/(p+1)} s^{(p+1-2\lambda)/2\lambda} \).

\[\square\]

**Remarks:** Although \( D(\mathcal{C}) \) is dense in \( V \), we cannot in general extend the result of Theorem 1.2 to \( y_0 \in V \); this is due to the fact that the constants \( c_1 \) and \( c_2 \) in (3.8) depend strongly on \( \sup_{t \geq 0} \| \Delta y(t) \|_V \). However, with more restrictive assumptions on the nonlinear term, we can obtain a priori estimates for which the constants do not depend on \( y_0 \in D(\mathcal{C}) \). More precisely, if \( g: \mathbb{C} \to \mathbb{C} \) satisfies:
\[
|g(z)| \leq C_1 |z|^{\lambda} \quad \text{if} \quad |z| \leq 1,
\]
\[
|g(z)| \leq C_2 |z| \quad \text{if} \quad |z| \geq 1,
\]
\[
g(z)^{\frac{p}{2}} \geq C_3 \min\{|z|^2, |z|^{p+1}\},
\]
then we can proceed as in Zuazua [10] and the decay rates in Theorem 1.2 may be extended by density to \( y_0 \in V \).
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